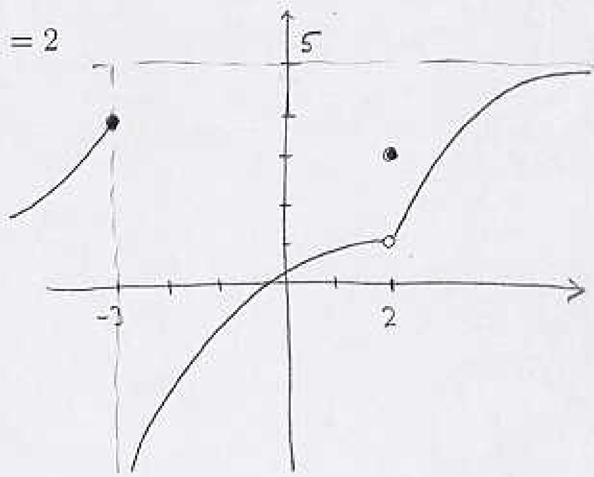
Fall '04/MAT 250/Exam 1 **Name:** Solution *Show all your work.*

1. (5pts) Sketch the graph of a function with the following properties:

f is defined on all of $\mathbf{R}$
f is continuous except at $x = -3$ and $x = 2$
$\lim_{x \to -3^+} f(x) = -\infty$
$\lim_{x \to -3^-} f(x) = 4$
$\lim_{x \to 2} f(x)$ exists
$\lim_{x \to \infty} f(x) = 5$



2. (7pts) Consider the equation $x^3 - 3x^2 + x - 5 = 0$.
a) Use the Intermediate Value Theorem to show that this equation has a solution in the interval [0,4].
b) Use your calculator to find an interval of length 0.01 that contains this root. Explain why the IVT will guarantee there is root in the interval that you found.

a) Let $f(x) = x^3 - 3x^2 + x - 5$

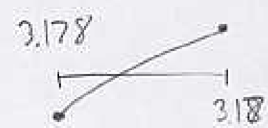
$f(0) = -5$

$f(4) = 15$

Since 0 is between -5 and 15, f is continuous (a polynomial), by IVT there is a number c between 0 and 4 so that $f(c) = 0$

b) Tracing gives us:

x	$f(x)$
3.178	-0.021
3.180	0.0106



Since 0 is between -0.021 and 0.0106 by IVT there is a number c between 3.178 and 3.180 so that $f(c) = 0$.

3.178 and 3.180 are 0.002 apart.

3. (13pts) Find the following limits algebraically.

a) $\lim\limits_{x\to 0} \dfrac{1-\sqrt{1-x^2}}{x} = \lim\limits_{x\to 0} \dfrac{1-\sqrt{1-x^2}}{x} \cdot \dfrac{1+\sqrt{1-x^2}}{1+\sqrt{1-x^2}} = \lim\limits_{x\to 0} \dfrac{1-(1-x^2)}{x(1+\sqrt{1-x^2})}$

$= \lim\limits_{x\to 0} \dfrac{\cancel{x^2}\ x}{\underset{1}{\cancel{x}}(1+\sqrt{1-x^2})} = \lim\limits_{x\to 0} \dfrac{x}{1+\sqrt{1-x^2}} = \dfrac{1}{1+\sqrt{1-0}} = \dfrac{1}{2}$

b) $\lim\limits_{x\to -\infty} x^4 - 5x^3 + x + 1 = \lim\limits_{x\to -\infty} x^4\left(1 - \underbrace{\dfrac{5}{x}}_{\to 0} + \underbrace{\dfrac{1}{x^3}}_{\to 0} + \underbrace{\dfrac{1}{x^4}}_{\to 0}\right) = \infty \cdot 1 = \infty$

c) $\lim\limits_{x\to 3+} \dfrac{4}{3-x} = \dfrac{4}{0-} = -\infty$

since $x \to 3+$

$x > 3$

so $3 - x < 0$

Thus $3 - x \to 0-$

"Belize"

4. (5pts) Use the theorem that rhymes with the name of a Central American country that starts with a B to find $\lim\limits_{x\to\infty} \dfrac{\cos x}{x^2+5}$.

$-1 \le \cos x \le 1 \qquad 1 \cdot \dfrac{1}{x^2+5}$

$-\dfrac{1}{x^2+5} \le \dfrac{\cos x}{x^2+5} \le \dfrac{1}{x^2+5}$

$\lim\limits_{x\to\infty} -\dfrac{1}{x^2+5} = -\dfrac{1}{\infty+5} = -\dfrac{1}{\infty} = 0$

$\lim\limits_{x\to\infty} \dfrac{1}{x^2+5} = \dfrac{1}{\infty+5} = \dfrac{1}{\infty} = 0$

By the squeeze theorem, $\lim\limits_{x\to\infty} \dfrac{\cos x}{x^2+5} = 0$

5. (9pts) If a ball is thrown into the air with velocity of 40 ft/s, its height in feet is given by $y = 40t - 16t^2$.
a) Find the velocity of the ball when $t = 2$. Is the ball moving up or down at $t = 2$? How can you tell?
b) At what height is the ball at $t = 2$? If it were to continue moving at the same velocity as at $t = 2$, how long (from $t = 2$) would it be until it hit the ground?
c) How long (from $t = 2$) until the ball actually hits the ground? Why is there a discrepancy with b)?

a) $v(2) = \lim_{t\to 2} \frac{y(t)-y(2)}{t-2} = \lim_{t\to 2} \frac{40t-16t^2-16}{t-2} = \lim_{t\to 2} \frac{-8(2t^2-5t+2)}{t-2}$

$= \lim_{t\to 2} \frac{-8(2t-1)\cancel{(t-2)}}{\cancel{(t-2)}} = -8(2\cdot 2-1) = -24$ ft/s

Ball is moving down since velocity is negative.

b) $y(2) = 16$ ft

It would take

$\frac{16}{24} = \frac{2}{3}$ s

c) $y = 0$

$40t - 16t^2 = 0$

$8t(5-2t) = 0$

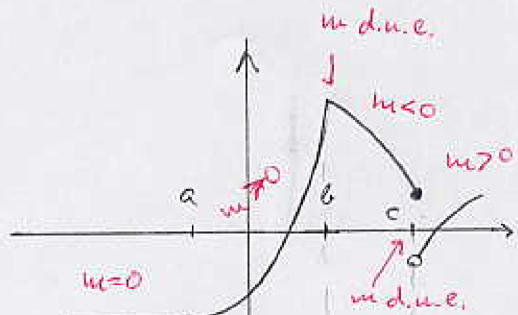
$t = 0, \frac{5}{2}$

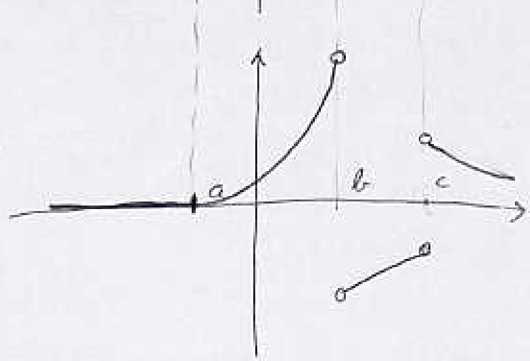
Ball hits ground at $t = 2.5$

It takes $\frac{1}{2}$ s for ball to hit the ground.

Shorter than b), because ball's velocity does not stay same – it increases as it falls.

6. (5pts) The graph of the function $f(x)$ is given. Use it to sketch the graph of $f'(x)$.





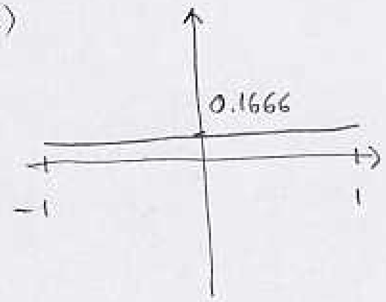
7. (6pts) Consider the function $f(x) = \dfrac{x - \sin x}{x^3}$.

a) Graph the function on your calculator (copy here) and use the graph to guess $\lim_{x\to 0} f(x)$.

b) Now enter some x's close to 0: 10^{-6}, 10^{-7}, 10^{-8}. What limit do these values suggest?

c) What do you think is going on in b)?

a)



It looks like

$$\lim_{x\to 0} \frac{x - \sin x}{x^3} = 0.1666$$

b)

x	$f(x)$
10^{-6}	0
10^{-7}	0
10^{-8}	0

This suggests

$$\lim_{x\to 0} f(x) = 0$$

c) When x is close to 0, x is very close to $\sin x$. Rounding error (or error in computing $\sin x$) gives $x - \sin x = 0$ in the calculator.

Bonus . (5pts) Show that $\dfrac{0}{0}$ is an indeterminate form. For that purpose, give two examples of functions f and g, which satisfy $\lim_{x\to 0} f(x) = 0$, $\lim_{x\to 0} g(x) = 0$ in both cases, but in one case $\lim_{x\to 0} \dfrac{f(x)}{g(x)} = 2$, and in the other case $\lim_{x\to 0} \dfrac{f(x)}{g(x)} = 0$. (Hint: think simple.)

Ex. 1

$f(x) = 2x$

$g(x) = x$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0$$

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{2\cancel{x}}{\cancel{x}} = 2$$

Ex. 2

$f(x) = x^2$

$g(x) = x$

$$\lim_{x\to 0} f(x) = 0$$

$$\lim_{x\to 0} g(x) = 0$$

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{\overset{x}{\cancel{x^2}}}{\underset{1}{\cancel{x}}} = \lim_{x\to 0} x = 0$$