

# Parameterizations of Univariate Orthogonal Wavelets with Short Support

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**Abstract.** In this note, we give a complete and simple parameterization for the length-six one-dimensional filters which satisfy the necessary conditions for orthogonality. This parameterization includes all compactly supported univariate scaling functions contained within the interval  $[0, 5]$  using dilation factor 2. These formulae are a convenient way to generate a continuum of wavelets with varying approximation properties ranging from scaling functions which only reproduce constants to ones which reproduce linears and quadratics. In addition, solutions for the length-eight and length-ten filters are given where the parameters have transcendental constraints. We conclude this note with some interesting numerical experiments comparing the parameterized wavelets of length six with the standard Haar, D4, and D6 wavelets in an image compression scheme.

## §1. Introduction

The use of compactly supported orthonormal wavelets is common in various applications. The Daubechies orthonormal wavelets are an important class which have been used extensively for applications such as image compression. The construction of wavelets usually begins with a scaling function  $\phi$  which satisfies various properties such as refinability and orthogonality. Numerous necessary and sufficient conditions have been developed to aid in the construction of wavelets (see [1,6], and many others). One method for constructing compactly supported scaling functions is to begin with the dilation equation

$$\phi(x) = \sum_k h_k \phi(2x - k)$$

with a finite number of coefficients. In order for a function of this type to be orthonormal to its integer shifts, the coefficients in the dilation equation must satisfy

$$m(1) = 1$$

$$|m(z)|^2 + |m(-z)|^2 = 1, \quad z = e^{i\omega},$$

where  $m(z) = \frac{1}{2} \sum_k h_k z^k$  is the associated trigonometric polynomial. For scaling functions with support contained in  $[0, 5]$  these necessary conditions suggest a simple technique for parameterizing their solution. A different approach in [8] gives the explicit parameterization of the length-four filters and a constructive technique for longer filters. The method shown here is the same technique we used to parameterize compactly supported bivariate scaling functions (see [2,3,4,5]). Another approach for dealing with the bivariate case can be seen in [7].

As will be shown, the necessary conditions for orthogonality yield a set of linear and nonlinear equations which necessarily imply that various sums of dilation coefficients form perfect squares. This leads to the introduction of the free parameters. In Section 2, we give the parameterization of the length-four filters for completeness. In Section 3, we derive the explicit parameterization for the length-six filters. Section 4 gives the constrained solutions for the length-eight and length-ten filters. Finally, we conclude in Section 5 with an interesting numerical experiment comparing the standard Haar, D4, and D6 wavelets with the parameterized wavelets of length-six in an image compression scheme.

## §2. Length-Four Solution

The formulas for the parameterization of the length-four filters were given in [8], but we include them here for completeness. Let  $H_4(z) = a_0 + b_0 z + a_1 z^2 + b_1 z^3$ .

**Lemma 1.**  $H_4(z)$  satisfies  $H_4(1) = 1$  and

$$|H_4(z)|^2 + |H_4(-z)|^2 = 1, \quad \forall z = e^{i\omega} \text{ and } \omega \in \mathbb{R}$$

if and only if

$$a_0 = \frac{1}{4} + \frac{1}{2\sqrt{2}} \cos \alpha, \quad b_0 = \frac{1}{4} + \frac{1}{2\sqrt{2}} \sin \alpha,$$

$$a_1 = \frac{1}{4} - \frac{1}{2\sqrt{2}} \cos \alpha, \quad b_1 = \frac{1}{4} - \frac{1}{2\sqrt{2}} \sin \alpha,$$

for any  $\alpha \in \mathbb{R}$ .

**Proof:** First, we have

$$|H_4(z)|^2 = a_0^2 + b_0^2 + a_1^2 + b_1^2$$

$$+ (a_0 b_0 + b_0 a_1 + a_1 b_1)(z + z^{-1})$$

$$+ (a_0 a_1 + b_0 b_1)(z^2 + z^{-2}) + a_0 b_1(z^3 + z^{-3}).$$

Then  $|H_4(z)|^2 + |H_4(-z)|^2 = 1$  implies that

$$\begin{aligned} a_0^2 + a_1^2 + b_0^2 + b_1^2 &= \frac{1}{2} \\ a_0 a_1 + b_0 b_1 &= 0. \end{aligned} \quad (1)$$

It follows from  $H_4(1) = 1$  that  $a_0 + a_1 + b_0 + b_1 = 1$ . Also,  $H(-1) = 0$  implies that  $a_0 + a_1 - b_0 - b_1 = 0$ . Thus, we have

$$a_0 + a_1 = b_0 + b_1 = \frac{1}{2}. \quad (2)$$

Moreover, the two nonlinear equations from (1) imply

$$(a_0 - a_1)^2 + (b_0 - b_1)^2 = \frac{1}{2}$$

or

$$a_0 - a_1 = \frac{1}{\sqrt{2}} \cos \alpha, \quad b_0 - b_1 = \frac{1}{\sqrt{2}} \sin \alpha.$$

Thus, after combining these equations with equation (2), we have the formulas as stated in the lemma.

On the other hand, using the expressions for the  $a_i$ 's and  $b_i$ 's with a straightforward calculation shows that in fact

$$|H_4(z)|^2 + |H_4(-z)|^2 = 1.$$

This completes the proof.  $\square$

**Example 1.** When  $\alpha = \frac{\pi}{4}$ , we get  $H_4(z) = \frac{1+z}{2}$ , which is associated with the Haar wavelet.

**Example 2.** When  $\alpha = \frac{5\pi}{12}$ , we get  $a_0 = \frac{1+\sqrt{3}}{8}$  and  $a_1 = \frac{3-\sqrt{3}}{8}$  as well as  $b_0 = \frac{3+\sqrt{3}}{8}$ ,  $b_1 = \frac{1-\sqrt{3}}{8}$ . Then  $H_4(z)$  is associated with the Daubechies D4 wavelet.

Next, we look for choices of  $\alpha$  where  $H_4(z)$  has a second-order vanishing moment. That is,  $H_4(z) = \left(\frac{1+z}{2}\right)^2 p(z)$  where  $p(z)$  is some trigonometric polynomial.

In this case  $\frac{d}{dz} H(z)|_{z=-1} = 0$  which is equivalent to  $b_0 - 2a_1 + 3b_1 = 0$ . That is,

$$\begin{aligned} 0 &= b_0 - 2a_1 + 3b_1 \\ &= \frac{1}{4} + \frac{1}{2\sqrt{2}} \sin \alpha - \frac{1}{2} + \frac{1}{\sqrt{2}} \cos \alpha + \frac{3}{4} - \frac{3}{2\sqrt{2}} \sin \alpha \\ &= \frac{1}{2} + \frac{1}{\sqrt{2}} (\cos \alpha - \sin \alpha) \\ &= \frac{1}{2} + \cos \left( \alpha + \frac{\pi}{4} \right), \end{aligned}$$

which is only satisfied by  $\alpha = \frac{5\pi}{12}$  or  $\alpha = \frac{13\pi}{12}$ . Both are associated with Daubechies' D4 wavelet. Thus, D4 is the only member of the family with two vanishing moments, which is a well-known fact.

### §3. Length-Six Solution

Let  $H_6(z) = a_0 + b_0z + a_1z^2 + b_1z^3 + a_2z^4 + b_2z^5$  be a trigonometric polynomial of  $z = e^{i\omega}$ .

**Lemma 2.**  $H_6(z)$  satisfies  $H_6(1) = 1$  and

$$|H_6(z)|^2 + |H_6(-z)|^2 = 1, \forall z = e^{i\omega}, \quad \omega \in \mathbb{R}$$

if and only if

$$\begin{aligned} a_0 &= \frac{1}{8} + \frac{1}{4\sqrt{2}} \cos \alpha + \frac{p}{2} \cos \beta \\ a_1 &= \frac{1}{4} - \frac{1}{2\sqrt{2}} \cos \alpha \\ a_2 &= \frac{1}{8} + \frac{1}{4\sqrt{2}} \cos \alpha - \frac{p}{2} \cos \beta \\ b_0 &= \frac{1}{8} + \frac{1}{4\sqrt{2}} \sin \alpha + \frac{p}{2} \sin \beta \\ b_1 &= \frac{1}{4} - \frac{1}{2\sqrt{2}} \sin \alpha \\ b_2 &= \frac{1}{8} + \frac{1}{4\sqrt{2}} \sin \alpha - \frac{p}{2} \sin \beta, \end{aligned}$$

where

$$p = \frac{1}{2} \sqrt{1 + \sin(\alpha + \frac{\pi}{4})}$$

for any  $\alpha, \beta \in \mathbb{R}$ .

**Proof:** First of all,  $H_6(1) = 1$  implies that  $\sum_{i=0}^2 a_i + b_i = 1$ . Also,  $H_6(-1) = 0$  implies that

$$\sum_{i=0}^2 a_i - \sum_{i=0}^2 b_i = 0.$$

It follows that

$$a_0 + a_1 + a_2 = b_0 + b_1 + b_2 = \frac{1}{2}. \quad (3)$$

Next,  $|H_6(z)|^2 + |H_6(-z)|^2 = 1$  implies that

$$\begin{aligned} a_0^2 + a_1^2 + a_2^2 + b_0^2 + b_1^2 + b_2^2 &= \frac{1}{2} \\ a_0a_1 + a_1a_2 + b_0b_1 + b_1b_2 &= 0 \\ a_0a_2 + b_0b_2 &= 0. \end{aligned} \quad (4)$$

From these nonlinear equations, it follows that

$$(a_0 - a_1 + a_2)^2 + (b_0 - b_1 + b_2)^2 = \frac{1}{2},$$

and

$$a_0 - a_1 + a_2 = \frac{1}{\sqrt{2}} \cos \alpha, \quad b_0 - b_1 + b_2 = \frac{1}{\sqrt{2}} \sin \alpha. \quad (5)$$

Combining (5) with (3), we have

$$\begin{aligned} a_1 &= \frac{1}{4} - \frac{1}{2\sqrt{2}} \cos \alpha, & b_1 &= \frac{1}{4} - \frac{1}{2\sqrt{2}} \sin \alpha \\ a_0 + a_2 &= \frac{1}{4} + \frac{1}{2\sqrt{2}} \cos \alpha, & b_0 + b_2 &= \frac{1}{4} + \frac{1}{2\sqrt{2}} \sin \alpha. \end{aligned} \quad (6)$$

Moreover,

$$\begin{aligned} (a_0 - a_2)^2 + (b_0 - b_2)^2 &= \frac{1}{2} - a_1^2 - b_1^2 \\ &= \frac{1}{2} - \left( \frac{1}{4} - \frac{1}{4\sqrt{2}} (\cos \alpha + \sin \alpha) \right)^2 \\ &= \frac{1}{4} + \frac{1}{4\sqrt{2}} (\cos \alpha + \sin \alpha) \\ &= \frac{1}{4} + \frac{1}{4} \sin \left( \alpha + \frac{\pi}{4} \right), \end{aligned}$$

which implies that

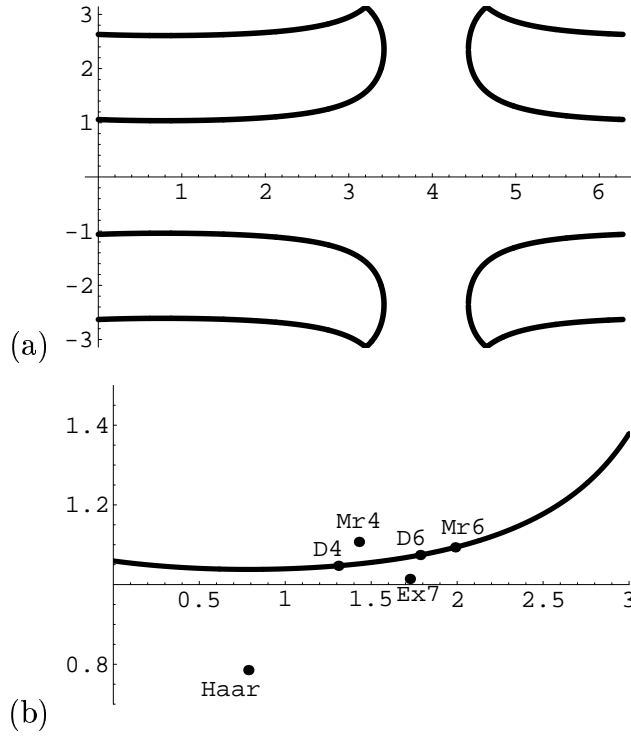
$$\begin{aligned} a_0 - a_2 &= \frac{1}{2} \sqrt{1 + \sin \left( \alpha + \frac{\pi}{4} \right)} \cos \beta \\ b_0 - b_2 &= \frac{1}{2} \sqrt{1 + \sin \left( \alpha + \frac{\pi}{4} \right)} \sin \beta. \end{aligned}$$

The result follows by adding these equations to those from (6).

On the other hand, a routine calculation establishes that  $H_6(z)$  with these coefficients satisfies  $H(1) = 1$  and  $|H_6(z)|^2 + |H_6(-z)|^2 = 1$ . This completes the proof.  $\square$

**Example 3.** If  $\alpha = \frac{\pi}{4}$  and  $\beta = \frac{\pi}{4}$ , then  $H_6(z)$  is associated with the Haar wavelet.

**Example 4.** If  $\alpha = \frac{5\pi}{12}$  and  $\beta = \frac{\pi}{3}$ , then  $H_6(z)$  is associated with the Daubechies D4 wavelet.



**Fig. 1.** (a) A plot of the ordered pairs  $(\alpha, \beta)$  for which  $H_6(z)$  has two vanishing moments, (b) a zoomed in plot of the above graph showing the positions of Haar, D4, D6, Example 7 (Ex7), and the most-regular length-four (Mr4) and length-six (Mr6) scaling functions.

**Example 5.** *If*

$$\cos \alpha = -\frac{1}{4}\sqrt{8 + \sqrt{15 + 12\sqrt{10}}}, \quad \sin \alpha = \frac{1}{4}\sqrt{8 - \sqrt{15 + 12\sqrt{10}}},$$

$$\cos \beta = \frac{1}{4}\sqrt{8 - 8\sqrt{-25 + 8\sqrt{10}}}, \quad \sin \beta = \frac{1}{4}\sqrt{8 + 8\sqrt{-25 + 8\sqrt{10}}},$$

then  $H_6(z)$  is the filter associated with Daubechies D6 wavelet.

**Example 6.** *To see when  $H_6(z)$  has two vanishing moments, we require  $H'(-1) = 0$  which implies*

$$\begin{aligned} 0 &= H'(-1) = b_0 - 2a_1 + 3b_1 - 4a_2 + 5b_2 \\ &= 1/2 + 2b_1 + 4b_2 - 2a_1 - 4a_2 \\ &= 1/2 - \sqrt{2(1 + \sin(\alpha + \pi/4))} \sin(\beta - \pi/4) \end{aligned}$$

or

$$\sin(\alpha + \pi) = \frac{1}{8 \sin^2(\beta - \pi/4)} - 1.$$

When the ordered pair  $(\alpha, \beta)$  lies along the curves given in Figure 1(a),  $H_6(z)$  has two vanishing moments.

**Example 7.** When  $\alpha = 1.725080699801023$  and  $\beta = 1.01424683724616$ , the scaling function is as shown in Figure 2(b). This scaling function performed better than Haar,  $D4$ , and  $D6$  in our numerical experiments as shown in the last Section. It should be noted that this scaling function does not have a second-order vanishing moment as can be seen in Figure 1(b).

**Example 8.** With  $\alpha = 1.4288992721907328$ ,  $\beta = 1.1071487177940904$ ,  $H_6(z)$  is associated with the most-regular length-four filter as given in [1]. When  $\alpha = 1.9886461158096038$  and  $\beta = 1.0934936891036087$ ,  $H_6(z)$  is associated with the most-regular length-six filter as given in [1]. Each of these scaling functions are shown in Figure 2(c) and 2(d).

#### §4. Constrained Solutions of Lengths Eight and Ten

The length-eight solution has 4 parameters and a transcendental constraint, and the length-ten solution has 5 parameters and a transcendental constraint as well. We will need the following lemma in the proofs of Lemmas 4 and 5.

**Lemma 3.** Suppose  $a, b, c$ , and  $d \in \mathbb{R}$ . Then  $a^2 + b^2 + c^2 + d^2 = 1$  if and only if

$$a = \cos \beta \cos \gamma, \quad b = \cos \beta \sin \gamma, \quad c = \sin \beta \cos \theta, \quad d = \sin \beta \sin \theta$$

for  $\beta, \gamma$ , and  $\theta \in \mathbb{R}$ .

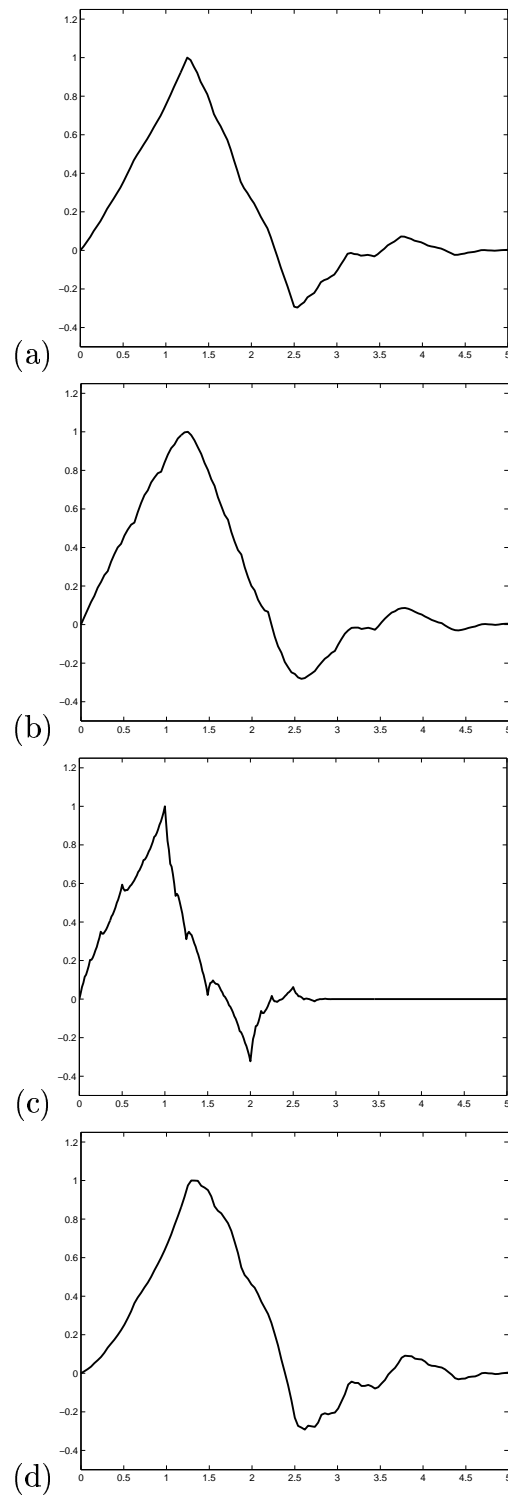
Let  $H_8(z) = a_0 + b_0z + a_1z^2 + b_1z^3 + a_2z^4 + b_2z^5 + a_3z^6 + b_3z^7$  be a trigonometric polynomial of  $z = e^{i\omega}$ .

**Lemma 4.**  $H_8(z)$  satisfies  $H_8(1) = 1$  and

$$|H_8(z)|^2 + |H_8(-z)|^2 = 1, \quad \forall z = e^{i\omega}, \quad \omega \in \mathbb{R}$$

if and only if

$$\begin{aligned} a_0 &= \frac{1}{8} + \frac{1}{4\sqrt{2}} \cos \alpha + \frac{1}{2\sqrt{2}} \cos \beta \cos \gamma \\ a_1 &= \frac{1}{8} - \frac{1}{4\sqrt{2}} \cos \alpha + \frac{1}{2\sqrt{2}} \cos \beta \sin \gamma \\ a_2 &= \frac{1}{8} + \frac{1}{4\sqrt{2}} \cos \alpha - \frac{1}{2\sqrt{2}} \cos \beta \cos \gamma \\ a_3 &= \frac{1}{8} - \frac{1}{4\sqrt{2}} \cos \alpha - \frac{1}{2\sqrt{2}} \cos \beta \sin \gamma \end{aligned}$$



**Fig. 2.** Graphs of four different scaling functions: (a) D6 scaling function with three vanishing moments from Example 5, (b) H6 scaling function from Example 7, (c) most-regular length-four scaling function from Example 8, (d) most-regular length-six scaling function from Example 8.

$$\begin{aligned}
 b_0 &= \frac{1}{8} + \frac{1}{4\sqrt{2}} \sin \alpha + \frac{1}{2\sqrt{2}} \sin \beta \cos \theta \\
 b_1 &= \frac{1}{8} - \frac{1}{4\sqrt{2}} \sin \alpha + \frac{1}{2\sqrt{2}} \sin \beta \sin \theta \\
 b_2 &= \frac{1}{8} + \frac{1}{4\sqrt{2}} \sin \alpha - \frac{1}{2\sqrt{2}} \sin \beta \cos \theta \\
 b_3 &= \frac{1}{8} - \frac{1}{4\sqrt{2}} \sin \alpha - \frac{1}{2\sqrt{2}} \sin \beta \sin \theta,
 \end{aligned}$$

where  $\alpha, \beta, \theta$ , and  $\gamma \in \mathbb{R}$  satisfy

$$\begin{aligned}
 &\sqrt{2} \cos \theta \sin \beta - 2 \cos \theta \sin \alpha \sin \beta + \sqrt{2} \cos \beta (\cos \gamma - \sin \gamma) \\
 &- 4 \cos^2 \beta \cos \gamma \sin \gamma - 2 \cos \alpha \cos \beta (\cos \gamma + \sin \gamma) \\
 &- \sqrt{2} \sin \beta \sin \theta - 2 \sin \alpha \sin \beta \sin \theta - 4 \cos \theta \sin^2 \beta \sin \theta = 0.
 \end{aligned}$$

**Proof:** As before,  $H_8(1) = 1$  implies that  $\sum_{i=0}^3 a_i + b_i = 1$  and  $H_8(-1) = 0$  implies that

$$\sum_{i=0}^3 a_i - \sum_{i=0}^3 b_i = 0.$$

It follows that

$$a_0 + a_1 + a_2 + a_3 = b_0 + b_1 + b_2 + b_3 = \frac{1}{2}. \quad (7)$$

Again,  $|H_8(z)|^2 + |H_8(-z)|^2 = 1$  implies the following set of nonlinear equations:

$$\begin{aligned}
 a_0^2 + a_1^2 + a_2^2 + a_3^2 + b_0^2 + b_1^2 + b_2^2 + b_3^2 &= \frac{1}{2} \\
 a_0 a_1 + a_1 a_2 + a_2 a_3 + b_0 b_1 + b_1 b_2 + b_2 b_3 &= 0 \\
 a_0 a_2 + a_1 a_3 + b_0 b_2 + b_1 b_3 &= 0 \\
 a_0 a_3 + b_0 b_3 &= 0.
 \end{aligned} \quad (8)$$

By adding all the equations together from (8), we see that

$$(a_0 - a_1 + a_2 - a_3)^2 + (b_0 - b_1 + b_2 - b_3)^2 = \frac{1}{2}.$$

So,

$$a_0 - a_1 + a_2 - a_3 = \frac{1}{\sqrt{2}} \cos \alpha, \quad b_0 - b_1 + b_2 - b_3 = \frac{1}{\sqrt{2}} \sin \alpha,$$

which implies that

$$\begin{aligned}
 a_0 + a_2 &= \frac{1}{4} + \frac{1}{2\sqrt{2}} \cos \alpha, & a_1 + a_3 &= \frac{1}{4} - \frac{1}{2\sqrt{2}} \cos \alpha \\
 b_0 + b_2 &= \frac{1}{4} + \frac{1}{2\sqrt{2}} \sin \alpha, & b_1 + b_3 &= \frac{1}{4} - \frac{1}{2\sqrt{2}} \sin \alpha.
 \end{aligned}$$

Continuing along this line, by subtracting the third equation from the first equation of (8), we have

$$(a_0 - a_2)^2 + (a_1 - a_3)^2 + (b_0 - b_2)^2 + (b_1 - b_3)^2 = \frac{1}{2}.$$

Using Lemma 3,

$$\begin{aligned} a_0 - a_2 &= \frac{1}{\sqrt{2}} \cos \beta \cos \gamma, & a_1 - a_3 &= \frac{1}{\sqrt{2}} \cos \beta \sin \gamma \\ b_0 - b_2 &= \frac{1}{\sqrt{2}} \sin \beta \cos \theta, & b_1 - b_3 &= \frac{1}{\sqrt{2}} \sin \beta \sin \theta. \end{aligned}$$

Combining these four equations with the previous group of four, we arrive at the necessary formulae as listed in the lemma.

Plugging these formulae into the nonlinear system of equations reveals that the first and third equations from (8) are satisfied, but the second and fourth equations produce a single transcendental constraint as stated in the lemma.  $\square$

**Example 9.** Let  $\alpha = 2.2400754386946162$ ,  $\beta = 0.7535419996522459$ ,  $\gamma = 0.9614024467911164$ , and  $\theta = -0.02541300114737489$ . Then  $H_8(z)$  is associated with Daubechies scaling function D8.

**Example 10.** In order for  $H_8(z)$  to have a second order vanishing moment, we require  $H'(-1) = 0$ . It follows that

$$a_1 + 2a_2 + 3a_3 = b_1 + 2b_2 + 3b_3 + \frac{1}{4}.$$

Using the solution formulae in Lemma 4, we have

$$\frac{1}{2} - \sin\left(\alpha - \frac{\pi}{4}\right) + 2 \cos \beta \sin\left(\gamma + \frac{\pi}{4}\right) - 2 \sin \beta \sin\left(\theta + \frac{\pi}{4}\right) = 0.$$

Combining this with the transcendental constraint in Lemma 4, we can find many solutions for  $H_8(z)$  with the second order vanishing moment. For example,

$$\alpha = \frac{5\pi}{12}, \beta = -\frac{\pi}{2}, \gamma = -\frac{\pi}{4}, \text{ and } \theta = -\frac{\pi}{4},$$

and

$$\alpha = \frac{5\pi}{12}, \beta = \pi, \gamma = -\frac{\pi}{4}, \text{ and } \theta = -\frac{\pi}{4}.$$

Let  $H_{10}(z) = a_0 + b_0z + a_1z^2 + b_1z^3 + a_2z^4 + b_2z^5 + a_3z^6 + b_3z^7 + a_4z^8 + b_4z^9$  be a trigonometric polynomial of  $z = e^{i\omega}$ .

**Lemma 5.**  $H_{10}(z)$  satisfies  $H_{10}(1) = 1$  and

$$|H_{10}(z)|^2 + |H_{10}(-z)|^2 = 1, \forall z = e^{i\omega}, \quad \omega \in \mathbb{R}$$

if and only if

$$\begin{aligned} a_0 &= \frac{1}{16} + \frac{1}{8\sqrt{2}} \cos \alpha + \frac{1}{4\sqrt{2}} \cos \beta \cos \gamma + \frac{r}{2} \cos \delta \\ a_1 &= \frac{1}{8} - \frac{1}{4\sqrt{2}} \cos \alpha + \frac{1}{2\sqrt{2}} \cos \beta \sin \gamma \\ a_2 &= \frac{1}{8} + \frac{1}{4\sqrt{2}} \cos \alpha - \frac{1}{2\sqrt{2}} \cos \beta \cos \gamma \\ a_3 &= \frac{1}{8} - \frac{1}{4\sqrt{2}} \cos \alpha - \frac{1}{2\sqrt{2}} \cos \beta \sin \gamma \\ a_4 &= \frac{1}{16} + \frac{1}{8\sqrt{2}} \cos \alpha + \frac{1}{4\sqrt{2}} \cos \beta \cos \gamma - \frac{r}{2} \cos \delta \\ b_0 &= \frac{1}{16} + \frac{1}{8\sqrt{2}} \sin \alpha + \frac{1}{4\sqrt{2}} \sin \beta \cos \theta + \frac{r}{2} \sin \delta \\ b_1 &= \frac{1}{8} - \frac{1}{4\sqrt{2}} \sin \alpha + \frac{1}{2\sqrt{2}} \sin \beta \sin \theta \\ b_2 &= \frac{1}{8} + \frac{1}{4\sqrt{2}} \sin \alpha - \frac{1}{2\sqrt{2}} \sin \beta \cos \theta \\ b_3 &= \frac{1}{8} - \frac{1}{4\sqrt{2}} \sin \alpha - \frac{1}{2\sqrt{2}} \sin \beta \sin \theta \\ b_4 &= \frac{1}{16} + \frac{1}{8\sqrt{2}} \sin \alpha + \frac{1}{4\sqrt{2}} \sin \beta \cos \theta - \frac{r}{2} \sin \delta, \end{aligned}$$

where

$$r = \sqrt{\frac{1}{2} - a_1^2 - a_2^2 - a_3^2 - b_1^2 - b_2^2 - b_3^2}$$

and  $\alpha, \beta, \theta, \gamma$ , and  $\delta \in \mathbb{R}$  satisfy

$$\begin{aligned} &\cos \beta \left( \cos \gamma (\sqrt{2} - 2 \cos \alpha) - 8\sqrt{2}r \cos \delta \sin \gamma \right) + \\ &\sin \beta \left( \cos \theta (\sqrt{2} - 2 \sin \alpha) - 8\sqrt{2}r \sin \delta \sin \theta \right) = 0. \end{aligned}$$

**Proof:** Along the same lines, we have the following nonlinear equations to solve:

$$\begin{aligned} a_0^2 + a_1^2 + a_2^2 + a_3^2 + a_4^2 + b_0^2 + b_1^2 + b_2^2 + b_3^2 + b_4^2 &= \frac{1}{2} \\ a_0a_1 + a_1a_2 + a_2a_3 + a_3a_4 + b_0b_1 + b_1b_2 + b_2b_3 + b_3b_4 &= 0 \\ a_0a_2 + a_1a_3 + a_2a_4 + b_0b_2 + b_1b_3 + b_2b_4 &= 0 \\ a_0a_3 + a_1a_4 + b_0b_3 + b_1b_4 &= 0 \\ a_0a_4 + b_0b_4 &= 0, \end{aligned} \tag{9}$$

as well as the linear equation  $\sum_{i=0}^4 a_i = \sum_{i=0}^4 b_i$ . Using various combinations of these equations as we have before, it can be seen that

$$\begin{aligned} a_0 + a_1 + a_2 + a_3 + a_4 &= \frac{1}{2}, & b_0 + b_1 + b_2 + b_3 + b_4 &= \frac{1}{2} \\ a_0 - a_1 + a_2 - a_3 + a_4 &= \frac{1}{\sqrt{2}} \cos \alpha, & b_0 - b_1 + b_2 - b_3 + b_4 &= \frac{1}{\sqrt{2}} \sin \alpha \\ a_0 - a_2 + a_4 &= \frac{1}{\sqrt{2}} \cos \beta \cos \gamma, & b_0 - b_2 + b_4 &= \frac{1}{\sqrt{2}} \sin \beta \cos \theta \\ a_1 - a_3 &= \frac{1}{\sqrt{2}} \cos \beta \sin \gamma, & b_1 - b_3 &= \frac{1}{\sqrt{2}} \sin \beta \sin \theta. \end{aligned}$$

So, we can solve for  $a_1, a_2, a_3, b_1, b_2,$  and  $b_3$  as

$$\begin{aligned} a_1 &= \frac{1}{8} - \frac{1}{4\sqrt{2}} \cos \alpha + \frac{1}{2\sqrt{2}} \cos \beta \sin \gamma \\ a_2 &= \frac{1}{8} + \frac{1}{4\sqrt{2}} \cos \alpha - \frac{1}{2\sqrt{2}} \cos \beta \cos \gamma \\ a_3 &= \frac{1}{8} - \frac{1}{4\sqrt{2}} \cos \alpha - \frac{1}{2\sqrt{2}} \cos \beta \sin \gamma \\ b_1 &= \frac{1}{8} - \frac{1}{4\sqrt{2}} \sin \alpha + \frac{1}{2\sqrt{2}} \sin \beta \sin \theta \\ b_2 &= \frac{1}{8} + \frac{1}{4\sqrt{2}} \sin \alpha - \frac{1}{2\sqrt{2}} \sin \beta \cos \theta \\ b_3 &= \frac{1}{8} - \frac{1}{4\sqrt{2}} \sin \alpha - \frac{1}{2\sqrt{2}} \sin \beta \sin \theta. \end{aligned}$$

Using the first and last equation from (9), we have

$$(a_0 - a_4)^2 + a_1^2 + a_2^2 + a_3^2 + (b_0 - b_4)^2 + b_1^2 + b_2^2 + b_3^2 = \frac{1}{2}.$$

Hence, it follows that

$$\begin{aligned} a_0 - a_4 &= r \cos \delta \\ b_0 - b_4 &= r \sin \delta \\ r &= \sqrt{\frac{1}{2} - a_1^2 - a_2^2 - a_3^2 - b_1^2 - b_2^2 - b_3^2}. \end{aligned}$$

Thus, the necessary formulae from the lemma follow. Upon substituting these formulae back into the nonlinear equations, we find that the first, third, and last equations are satisfied where as the second and fourth produce a single transcendental equation.  $\square$

**Example 11.** Let  $\alpha = 2.6829477415207257$ ,  $\beta = 0.714939482206344$ ,  $\gamma = 1.028362000753886$ ,  $\theta = 0.2225085841811395$ , and suppose  $\delta = 1.2907830472783552$ . Then  $H_{10}(z)$  is associated with the Daubechies D10 wavelet.

**Example 12.** In order for  $H_{10}(z)$  to satisfy the second order vanishing moment condition, we need  $H'_{10}(-1) = 0$ , i.e.

$$\begin{aligned} H'_{10}(-1) &= b_0 + 3b_1 + 5b_2 + 7b_3 + 9b_4 - 2a_1 - 4a_2 - 6a_3 - 8a_4 \\ &= \frac{1}{2} + 4r \cos \delta - 4r \sin \delta + \sqrt{2} \cos \beta \sin \gamma - \sqrt{2} \sin \beta \sin \theta \\ &= 0. \end{aligned}$$

There are numerous possibilities. For example, when

$$\alpha = 2.8975168508124955, \beta = \frac{3\pi}{4}, \gamma = \frac{\pi}{6}, \theta = 0, \text{ and } \delta = \frac{\pi}{4},$$

then both the transcendental condition and the second moment condition are satisfied.

### §5. Numerical Experiment

We have implemented an image compression scheme as a means of comparing the parameterized solutions of length six with the standard wavelets of Haar, D4, and D6. (We are still working on the comparison using  $H_8$  and  $H_{10}$ . The results will be reported elsewhere.) The scheme consists of the following:

- Decomposing the gray-scale values of various images with size  $512 \times 512$  to a maximum number of levels.
- Encoding the decomposed image using an embedded zero-tree encoder to a specified file size. All of the images in this experiment have a file size of 262,159 bytes which is approximately one byte per pixel. The actual compressed file sizes are 32,793 bytes (8:1), 16,409 bytes (16:1), 8,217 bytes (32:1), and 4,121 bytes (64:1).
- Decoding the compressed file.
- Reconstructing the image using the wavelet transform and rounding the values to the nearest integer.
- Calculating the peak signal to noise ratio (PSNR) which is a measure of the root mean squared error (RMSE) in the sense that

$$\begin{aligned} RMSE &= \frac{1}{512^2} \sum_{i,j=1}^{512} (p_{i,j} - \hat{p}_{i,j})^2 \\ PSNR &= 20 \log_{10} \left( \frac{255}{RMSE} \right) \end{aligned}$$

where  $p_{i,j}$  is the original gray-scale value and  $\hat{p}_{i,j}$  is the value after reconstruction.

| Image: Lena $512 \times 512$         |         |         |         |         |
|--------------------------------------|---------|---------|---------|---------|
| Wavelet                              | 8:1     | 16:1    | 32:1    | 64:1    |
| Haar                                 | 36.2258 | 32.6462 | 29.5685 | 27.5420 |
| D4                                   | 38.4440 | 34.9209 | 31.6733 | 28.8185 |
| D6                                   | 38.7819 | 35.3234 | 32.0479 | 29.0727 |
| H6(Ex7)                              | 38.8167 | 35.4208 | 32.1585 | 29.1588 |
| Image: Barbara $512 \times 512$      |         |         |         |         |
| Wavelet                              | 8:1     | 16:1    | 32:1    | 64:1    |
| Haar                                 | 30.4954 | 26.8119 | 24.6329 | 22.7409 |
| D4                                   | 32.8675 | 28.6364 | 25.6853 | 23.3821 |
| D6                                   | 33.4311 | 29.0735 | 25.9431 | 23.4715 |
| H6(Ex7)                              | 33.6441 | 29.2804 | 26.1074 | 23.5789 |
| Image: Boat $512 \times 512$         |         |         |         |         |
| Wavelet                              | 8:1     | 16:1    | 32:1    | 64:1    |
| Haar                                 | 34.7720 | 30.7103 | 27.5762 | 25.4130 |
| D4                                   | 35.6517 | 31.5910 | 28.5165 | 26.0746 |
| D6                                   | 35.8593 | 31.8088 | 28.6402 | 26.1880 |
| H6(Ex7)                              | 35.9301 | 31.9080 | 28.7187 | 26.2733 |
| Image: Finger-print $512 \times 512$ |         |         |         |         |
| Wavelet                              | 8:1     | 16:1    | 32:1    | 64:1    |
| Haar                                 | 32.4415 | 29.9961 | 28.5828 | 27.3396 |
| D4                                   | 33.3224 | 30.8632 | 29.4448 | 27.9762 |
| D6                                   | 33.8077 | 31.2097 | 29.8049 | 28.2562 |
| H6(Ex7)                              | 33.9236 | 31.3387 | 29.9687 | 28.3928 |
| Image: Marmousi $512 \times 512$     |         |         |         |         |
| Wavelet                              | 8:1     | 16:1    | 32:1    | 64:1    |
| Haar                                 | 35.7244 | 31.1309 | 27.7014 | 25.0277 |
| D4                                   | 41.3471 | 36.2831 | 31.8262 | 27.5415 |
| D6                                   | 43.6310 | 37.7401 | 33.0496 | 28.9160 |
| H6(Ex7)                              | 45.5190 | 38.6440 | 33.9109 | 29.5047 |
| Image: Crowd $512 \times 512$        |         |         |         |         |
| Wavelet                              | 8:1     | 16:1    | 32:1    | 64:1    |
| Haar                                 | 33.3109 | 29.2032 | 26.1346 | 23.7222 |
| D4                                   | 34.7214 | 30.6307 | 27.4101 | 24.8170 |
| D6                                   | 35.1740 | 31.0444 | 27.7601 | 25.0348 |
| H6(Ex7)                              | 35.2893 | 31.1590 | 27.8690 | 25.1465 |

**Tab. 1.** PSNR comparison of the standard Daubechies wavelets and the parameterized wavelet of length six from Example 7.

Figure 3 shows the images used in the compression scheme, and Tab. 1 gives the PSNR values for the standard wavelets versus a parameterized wavelet for each image at various compression ratios. The parameterized wavelet was selected based upon having the largest PSNR values for the parameter values near D6. The search for the parameterized wavelet was limited, and a more exhaustive search may yield better results. Note that a higher PSNR value is better. In this case, we were able to find a single



**Fig. 3.** Test images: Lenna, Finger-print, Barbara, Marmousi, Boat, and Crowd.

length six parameterized filter given in Example 7 which out-performed all the others tested on all six test images and at each compression ratio.

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