The nonexistence of bivariate symmetric wavelets with short support and two vanishing moments

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Abstract. In this note, we show the nonexistence of bivariate symmetric wavelets with support size $[0,7]\times[0,7]$ with more than one vanishing moment. We derive a necessary solution for the first order vanishing moment condition, and we show that this solution is incompatible for higher vanishing moments.

§1. Introduction

The most common wavelets used for image processing are the tensor-product of univariate compactly supported orthonormal wavelets. Of this class of wavelets, only the Haar wavelet is symmetric which gives its associated filter the property of linear phase. Since Daubechies' work[5], numerous generalizations of wavelets have been developed including biorthogonal wavelets, multiwavelets, and bivariate wavelets. Since 1992, several examples of bivariate compactly supported orthonormal and biorthogonal wavelets have been constructed. See Cohen and Daubechies'93[3] for nonseparable bidimensional wavelets, J. Kovačević and M. Vetterli'92[8] for nonseparable filters and wavelets based on a generalized dilation matrix, He and Lai'97[6] for the complete solution of bivariate compactly supported wavelets with filter size up to 4×4, Belogay and Wang'99[2] for a special construction of bivariate nonseparable wavelets for any given regularity, and Ayache'99[1] for nonseparable dyadic compactly supported wavelets with arbitrary regularity. See also Cohen and Schlenker'93[4], Riemenschneider and Shen'97[10], and He and Lai'98[7] for bivariate biorthogonal box spline wavelets.

It is well-known that in the univariate setting, there does not exist symmetric compactly supported orthonormal wavelets except Haar for dilation factor 2. We are interested in the construction of symmetric wavelets in the

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bivariate setting with dilation matrix $2I$ which have compact support and vanishing moments. We start with a scaling function ϕ. Let

$$\hat{\phi}(\omega_1,\omega_2)=\prod_{k=1}^{\infty} m(e^{\omega_1/2^k},e^{\omega_2/2^k})$$

be the Fourier transform of ϕ, where

$$m(x,y)=\sum_{j=0}^{N}\sum_{k=0}^{N} c_{jk}x^j y^k$$

is a trigonometric polynomial satisfying $m(1,1)=1$. In addition, the trigonometric polynomial $m(x,y)$ satisfies the orthonormality condition

$$|m(x,y)|^2+|m(-x,y)|^2+|m(x,-y)|^2+|m(-x,-y)|^2=1.$$

Let $\psi_i(x,y)$ be the corresponding wavelet satisfying

$$\hat{\psi}_i(\omega_1,\omega_2)=m_i(e^{\omega_1/2},e^{\omega_2/2})\hat{\phi}(\omega_1/2,\omega_2/2), i=1,2,3,$$

where the m_i are trigonometric polynomials such that the following matrix

$$\begin{bmatrix} m(x,y) & m(-x,y) & m(x,-y) & m(-x,-y)\\ m_1(x,y) & m_1(-x,y) & m_1(x,-y) & m_1(-x,-y)\\ m_2(x,y) & m_2(-x,y) & m_2(x,-y) & m_2(-x,-y)\\ m_3(x,y) & m_3(-x,y) & m_3(x,-y) & m_3(-x,-y)\end{bmatrix}$$

is unitary. Moreover, we are interested in symmetric scaling functions ϕ with a certain number of vanishing moments in the sense that their associated trigonometric polynomial $m(x,y)$ satisfies

$$m(1/x,1/y)=x^{-N}y^{-N}m(x,y)$$

as well as

$$\frac{\partial^k}{\partial x^k}m(x,y)\bigg|_{x=-1}=\frac{\partial^k}{\partial y^k}m(x,y)\bigg|_{y=-1}=0,\qquad 0\le k\le M-1.$$

The symmetry condition provides ϕ with the property of linear phase and the vanishing moment conditions provide ϕ with polynomial reproduction up to degree $M-1$. polynomials If $m(x,y)$ satisfies the symmetric property, then the associated wavelets can easily be found by using $\hat{\phi}$ and

$$m_1(x,y)=m(-x,y),\ \ m_2(x,y)=x\cdot m(x,-y),\ \ \text{and}\ \ m_3(x,y)=x\cdot m(-x,-y).$$

In summary, we are looking for trigonometric polynomials $m(x,y)=\sum_{j=0}^{N}\sum_{k=0}^{N} c_{jk}x^j y^k$ which satisfy the following properties:

(i) Existence: $m(1,1)=1$.
(ii) Orthogonality:$|m(x,y)|^2+|m(-x,y)|^2+|m(x,-y)|^2+|m(-x,-y)|^2=1$.
(iii) Symmetry: $m(1/x,1/y)=x^{-N}y^{-N}m(x,y)$.
(iv) M vanishing moments: $m(x,y)=(x+1)^M(y+1)^M\tilde{m}(x,y)$ where $\tilde{m}(x,y)$ is another trigonometric polynomial.

Recently, the authors have successfully constructed the complete solution of all bivariate nonseparable compactly supported symmetric wavelets with one vanishing moment and support $[0,5]\times[0,5]$ in Lai and Roach'99[9]. That is, for $N=5$ and $M=1$, we construct a two-parameter family of $m(x,y)$ which constitutes a scaling function and the corresponding wavelets. However, we were unable to find scaling functions and wavelets with $N=5$ and $M=2$.

In this note, we develop a necessary condition for the trigonometric polynomials $m(x,y)$ which satisfy the symmetry condition, the vanishing moment condition, and the orthonormality condition for $N=7$ and $M=1$. We then show that there are no trigonometric polynomials for $N=7$ and $M=2$, and consequently no symmetric bivariate scaling functions with two vanishing moments for the support size we are considering.

To be more precise, write $m(x,y)=\sum_{i=0}^{7}\sum_{j=0}^{7}h_{ij}x^iy^j$. Let us express $m(x,y)$ in polyphase form as

$$m(x,y)=f_0(x^2,y^2)+xf_1(x^2,y^2)+yf_2(x^2,y^2)+xyf_3(x^2,y^2)$$

with

$$f_0(x,y)=\sum_{i=0}^{3}\sum_{j=0}^{3}a_{ij}x^iy^j, \quad = \quad f_1(x,y) \quad = \quad \sum_{i=0}^{3}\sum_{j=0}^{3}b_{ij}x^iy^j$$
$$f_2(x,y)=\sum_{i=0}^{3}\sum_{j=0}^{3}c_{ij}x^iy^j, \quad f_3(x,y) \quad = \quad \sum_{i=0}^{3}\sum_{j=0}^{3}d_{ij}x^iy^j.$$

By (iii), we have

$$m(x,y)=\begin{bmatrix}1\\x\\x^2\\x^3\\x^4\\x^5\\x^6\\x^7\end{bmatrix}^T\begin{bmatrix}a_0&b_0&a_1&b_1&a_2&b_2&a_3&b_3\\b_{15}&a_{15}&b_{14}&a_{14}&b_{13}&a_{13}&b_{12}&a_{12}\\a_4&b_4&a_5&b_5&a_6&b_6&a_7&b_7\\b_{11}&a_{11}&b_{10}&a_{10}&b_9&a_9&b_8&a_8\\a_8&b_8&a_9&b_9&a_{10}&b_{10}&a_{11}&b_{11}\\b_7&a_7&b_6&a_6&b_5&a_5&b_4&a_4\\a_{12}&b_{12}&a_{13}&b_{13}&a_{14}&b_{14}&a_{15}&b_{15}\\b_3&a_3&b_2&a_2&b_1&a_1&b_0&a_0\end{bmatrix}\begin{bmatrix}1\\y\\y^2\\y^3\\y^4\\y^5\\y^6\\y^7\end{bmatrix}.(1)$$

We will derive a necessary condition on these a_i's and b_i's such that $m(x,y)$ satisfies (ii). Then, we show that the a_i's and b_i's do not satisfy the second order vanishing moment condition.

The paper is organized as follows: We first give the linear and nonlinear equations arising from (i), (ii), (iii) and (iv) with $N = 7$ and $M = 1$ in Section 2. We next analyze these linear and nonlinear equations to find necessary conditions for the a_i's and b_i's in section 3. In Section 4, we express the second order vanishing moment condition (iv) with $M = 2$ as a system of linear equations. We conclude section 4 by showing that the necessary conditions from section 3 are incompatible with the second order vanishing moment condition from section 4.

§2. Linear and Nonlinear Equations

Properties (i) and (iii) imply that

$$\sum_{i=0}^{15} \sum a_i = \frac{1}{2} \tag{2}$$

where $\sum a_i := a_i + b_i$ for simplicity.

Property (ii) implies the following 25 nonlinear equations:

$$\sum a_0 a_{15} = 0 \tag{3}$$

$$\sum a_3 a_{12} = 0 \tag{4}$$

$$\sum a_0 a_{11} + a_4 a_{15} = 0 \tag{5}$$

$$\sum a_0 a_{14} + a_1 a_{15} = 0 \tag{6}$$

$$\sum a_2 a_{12} + a_3 a_{13} = 0 \tag{7}$$

$$\sum a_3 a_8 + a_7 a_{12} = 0 \tag{8}$$

$$\sum a_0 a_7 + a_4 a_{11} + a_8 a_{15} = 0 \tag{9}$$

$$\sum a_0 a_{13} + a_1 a_{14} + a_2 a_{15} = 0 \tag{10}$$

$$\sum a_1 a_{12} + a_2 a_{13} + a_3 a_{14} = 0 \tag{11}$$

$$\sum a_3 a_4 + a_7 a_8 + a_{11} a_{12} = 0 \tag{12}$$

$$\sum a_0 a_3 + a_4 a_7 + a_8 a_{11} + a_{12} a_{15} = 0 \tag{13}$$

$$\sum a_0 a_{10} + a_1 a_{11} + a_4 a_{14} + a_5 a_{15} = 0 \tag{14}$$

$$\sum a_0 a_{12} + a_1 a_{13} + a_2 a_{14} + a_3 a_{15} = 0 \tag{15}$$

$$\sum a_2 a_8 + a_3 a_9 + a_6 a_{12} + a_7 a_{13} = 0 \tag{16}$$

$$\sum a_0 a_6 + a_1 a_7 + a_4 a_{10} + a_5 a_{11} + a_8 a_{14} + a_9 a_{15} = 0 = \tag{17}$$

$$\sum a_0 a_9 + a_1 a_{10} + a_2 a_{11} + a_4 a_{13} + a_5 a_{14} + a_6 a_{15} = 0 = \tag{18}$$

$$\sum a_1a_8 + a_2a_9 + a_3a_{10} + a_5a_{12} + a_6a_{13} + a_7a_{14} = 0 = \tag{19}$$

$$\sum a_2a_4 + a_3a_5 + a_6a_8 + a_7a_9 + a_{10}a_{12} + a_{11}a_{13} = 0 = \tag{20}$$

$$\sum = a_0a_8 + a_1a_9 + a_2a_{10} + a_3a_{11} + a_4a_{12} + a_5a_{13} + a_6a_{14} + a_7a_{15} = 0 \tag{21}$$

$$\sum a_0a_2 + a_1a_3 + a_4a_6 + a_5a_7 + a_8a_{10} + a_9a_{11} + a_{12}a_{14} + a_{13}a_{15} = 0 \tag{22}$$

$$\sum a_0a_5 + a_1a_6 + a_2a_7 + a_4a_9 + a_5a_{10} + a_6a_{11} + a_8a_{13} + a_9a_{14} + a_{10}a_{15} = 0 \tag{23}$$

$$\sum a_1a_4 + a_2a_5 + a_3a_6 + a_5a_8 + a_6a_9 + a_7a_{10} + a_9a_{12} + a_{10}a_{13} + a_{11}a_{14} = 0 \tag{24}$$

$$\sum a_0a_1 + a_1a_2 + a_2a_3 + a_4a_5 + a_5a_6 + a_6a_7 + a_8a_9 + a_9a_{10} + a_{10}a_{11} + a_{12}a_{13} + a_{13}a_{14} + a_{14}a_{15} = 0 \quad = (25)$$

$$\sum a_0a_4 + a_1a_5 + a_2a_6 + a_3a_7 + a_4a_8 + a_5a_9 + a_6a_{10} + a_7a_{11} + a_8a_{12} + a_9a_{13} + a_{10}a_{14} + a_{11}a_{15} = 0 \tag{26}$$

$$\sum_{i=0}^{15} \sum a_i^2 = \frac{1}{8}. \tag{27}$$

Property (iv) with $M = 1$ implies

$$a_0 + a_1 + a_2 + a_3 = b_0 + b_1 + b_2 + b_3 \tag{28}$$
$$a_4 + a_5 + a_6 + a_7 = b_4 + b_5 + b_6 + b_7 \tag{29}$$
$$a_8 + a_9 + a_{10} + a_{11} = b_8 + b_9 + b_{10} + b_{11} \tag{30}$$
$$a_{12} + a_{13} + a_{14} + a_{15} = b_{12} + b_{13} + b_{14} + b_{15} \tag{31}$$
$$a_0 + a_4 + a_8 + a_{12} = b_3 + b_7 + b_{11} + b_{15} \tag{32}$$
$$a_1 + a_5 + a_9 + a_{13} = b_2 + b_6 + b_{10} + b_{14} \tag{33}$$
$$a_2 + a_6 + a_{10} + a_{14} = b_1 + b_5 + b_9 + b_{13} \tag{34}$$
$$a_3 + a_7 + a_{11} + a_{15} = b_0 + b_4 + b_8 + b_{12}. \tag{35}$$

Using (28)-(31) and (2), we immediately get

$$\sum_{i=0}^{15} a_i = \sum_{i=0}^{15} b_i = 1/4. \tag{36}$$

§3. Necessary Conditions

We first group some nonlinear equations together to make perfect squares. This enables us to introduce parameters in order to simplify these equations.

Using (27), (26), (21), and (15), we have

$$\sum (a_0 + a_4 + a_8 + a_{12})^2 + (a_1 + a_5 + a_9 + a_{13})^2$$
$$+ (a_2 + a_6 + a_{10} + a_{14})^2 + (a_3 + a_7 + a_{11} + a_{15})^2 = \frac{1}{= 8}. \tag{37}$$

Moreover, using (6), (7), (14), (16), (17), (20), (22), we get

$$\sum (a_0 + a_4 + a_8 + a_{12})(a_2 + a_6 + a_{10} + a_{14})$$
$$+ (a_1 + a_5 + a_9 + a_{13})(a_3 + a_7 + a_{11} + a_{15}) = 0. \tag{38}$$

After using (32)-(35), (37), and (38), we obtain

$$((a_0 + a_4 + a_8 + a_{12}) \pm (a_2 + a_6 + a_{10} + a_{14}))^2$$
$$+ ((a_1 + a_5 + a_9 + a_{13}) \pm (a_3 + a_7 + a_{11} + a_{15}))^2 = \frac{1}{8}. \tag{39}$$

Choosing the plus sign in equation (39) and using (36) yields

$$a_0 + a_4 + a_8 + a_{12} + a_2 + a_6 + a_{10} + a_{14} = r_0,$$
$$a_1 + a_5 + a_9 + a_{13} + a_3 + a_7 + a_{11} + a_{15} = s_0,$$

where $r_0 + s_0 = 1/4$ and $r_0 s_0 = 0$. Giving us four cases: $r_0, s_0 = 1/4$ or 0.

Similarly, using (27), (25), (22), and (13), we get

$$\sum (a_0 + a_1 + a_2 + a_3)^2 + (a_4 + a_5 + a_6 + a_7)^2$$
$$+ (a_8 + a_9 + a_{10} + a_{11})^2 + (a_{12} + a_{13} + a_{14} + a_{15})^2 = \frac{1}{8}. \tag{40}$$

Using (21), (19), (18), (16), (14), (8), (5), we have

$$\sum (a_0 + a_1 + a_2 + a_3)(a_8 + a_9 + a_{10} + a_{11})$$
$$+ (a_4 + a_5 + a_6 + a_7)(a_{12} + a_{13} + a_{14} + a_{15}) = 0. \tag{41}$$

In a similar fashion as before, we have

$$a_0 + a_1 + a_2 + a_3 + a_8 + a_9 + a_{10} + a_{11} = t_0,$$
$$a_4 + a_5 + a_6 + a_7 + a_{12} + a_{13} + a_{14} + a_{15} = u_0,$$

where $t_0 + u_0 = 1/4$ and $t_0 u_0 = 0$.

Next using (21), (22), (14), and (16), we have

$$\sum (a_0 + a_2 + a_8 + a_{10})^2 + (a_1 + a_3 + a_9 + a_{11})^2$$
$$+ (a_4 + a_6 + a_{12} + a_{14})^2 + (a_5 + a_7 + a_{13} + a_{15})^2 = \frac{1}{8}. \tag{42}$$

Using (3), (4), (9)-(12), (23), and (24), we get

$$\sum (a_0 + a_2 + a_8 + a_{10})(a_5 + a_7 + a_{13} + a_{15}) + (a_4 + a_6 + a_{12} + a_{14})(a_1 + a_3 + a_9 + a_{11}) = 0. \tag{43}$$

It follows that

$$\sum ((a_0 + a_2 + a_8 + a_{10}) \pm (a_5 + a_7 + a_{13} + a_{15}))^2 + ((a_4 + a_6 + a_{12} + a_{14}) \pm (a_1 + a_3 + a_9 + a_{11}))^2 = \frac{1}{8}. \tag{44}$$

It should be noted that equation (44) requires the $\sum$ notation since there are no linear equations relating these sums of the a_i's to the sums of the b_i's. Upon choosing the plus sign, we can still use (36) to simplify (44). Denote $\hat{a} = a_0+a_2+a_8+a_{10}+a_5+a_7+a_{13}+a_{15}$ and $\hat{b} = b_0+b_2+b_8+b_{10}+b_5+b_7+b_{13}+b_{15}$, then (44) becomes

$$\hat{a}^2 + (\frac{1}{4} - \hat{a})^2 + \hat{b}^2 + (\frac{1}{4} - \hat{b})^2 = \frac{1}{8} \tag{45}$$

$$(\hat{a} - \frac{1}{8})^2 + (\hat{b} - \frac{1}{8})^2 = \frac{1}{32}. \tag{46}$$

Equation (46) can be solved in terms of a free parameter as

$$a_0 + a_2 + a_8 + a_{10} + a_5 + a_7 + a_{13} + a_{15} = p_1$$
$$b_0 + b_2 + b_8 + b_{10} + b_5 + b_7 + b_{13} + b_{15} = q_1,$$

where $p_1 = \frac{1}{8} + \frac{1}{4\sqrt{2}} \cos\gamma$ and $q_1 = \frac{1}{8} + \frac{1}{4\sqrt{2}} \sin\gamma$. We summarize the above information in the following lemma.

Lemma 1.

$$a_0 + a_2 + a_4 + a_6 + a_8 + a_{10} + a_{12} + a_{14} = r_0, \tag{47}$$
$$a_1 + a_3 + a_5 + a_7 + a_9 + a_{11} + a_{13} + a_{15} = s_0, \tag{48}$$
$$a_0 + a_1 + a_2 + a_3 + a_8 + a_9 + a_{10} + a_{11} = t_0, \tag{49}$$
$$a_4 + a_5 + a_6 + a_7 + a_{12} + a_{13} + a_{14} + a_{15} = u_0, \tag{50}$$
$$a_0 + a_2 + a_5 + a_7 + a_8 + a_{10} + a_{13} + a_{15} = p_1,, \tag{51}$$
$$a_1 + a_3 + a_4 + a_6 + a_9 + a_{11} + a_{12} + a_{14} = \frac{1}{4} - p_1, \tag{52}$$
$$b_0 + b_2 + b_4 + b_6 + b_8 + b_{10} + b_{12} + b_{14} = s_0, \tag{53}$$
$$b_1 + b_3 + b_5 + b_7 + b_9 + b_{11} + b_{13} + b_{15} = r_0, \tag{54}$$
$$b_0 + b_1 + b_2 + b_3 + b_8 + b_9 + b_{10} + b_{11} = t_0, \tag{55}$$
$$b_4 + b_5 + b_6 + b_7 + b_{12} + b_{13} + b_{14} + b_{15} = u_0, \tag{56}$$
$$b_0 + b_2 + b_5 + b_7 + b_8 + b_{10} + b_{13} + b_{15} = q_1, \tag{57}$$
$$b_1 + b_3 + b_4 + b_6 + b_9 + b_{11} + b_{12} + b_{14} = \frac{1}{4} - q_1., \tag{58}$$

where $r_0 + s_0 = \frac{1}{4}$, $r_0 s_0 = 0$, $t_0 + u_0 = \frac{1}{4}$, $t_0 u_0 = 0$, $p_1 = \frac{1}{8} + \frac{1}{4\sqrt{2}} \cos\gamma$, *and* $q_1 = \frac{1}{8} + \frac{1}{4\sqrt{2}} \sin\gamma$.

Here, we note that we have used (28)-(35) to get the equations for the sums of the b_i's.

We now refine our solutions to sums of four coefficients. Choosing the minus sign in equation (39) yields

$$a_0 + a_4 + a_8 + a_{12} = \frac{1}{2}(r_0 + r_1), \quad a_2 + a_6 + a_{10} + a_{14} = \frac{1}{2}(r_0 - r_1)$$
$$a_1 + a_5 + a_9 + a_{13} = \frac{1}{2}(s_0 + s_1), \quad a_3 + a_7 + a_{11} + a_{15} = \frac{1}{2}(s_0 - s_1),$$

where $r_1 = \frac{1}{4}\cos\alpha$ and $s_1 = \frac{1}{4}\sin\alpha$. Furthermore, equations (3), (4), (5), (8), (9), (12), and (13) yield

$$\sum (a_0 + a_4 + a_8 + a_{12})(a_3 + a_7 + a_{11} + a_{15}) = 0. \tag{59}$$

This together with (32) and (35), give the following constraint on our parameters

$$2(r_0 + r_1)(s_0 - s_1) = 0. \tag{60}$$

Because of the relationship between r_1 and s_1, equation (60) produces three cases: $r_1 = r_0$ and $s_1 = s_0$, $r_1 = -r_0$ and $s_1 = s_0$, or $r_1 = -r_0$ and $s_1 = -s_0$.

Similarly,

$$a_0 + a_1 + a_2 + a_3 = \frac{1}{2}(t_0 + t_1), \quad a_8 + a_9 + a_{10} + a_{11} = \frac{1}{2}(t_0 - t_1)$$
$$a_4 + a_5 + a_6 + a_7 = \frac{1}{2}(u_0 + u_1), \quad a_{12} + a_{13} + a_{14} + a_{15} = \frac{1}{2}(u_0 - u_1),$$

where $t_1 = \frac{1}{4}\cos\beta$ and $u_1 = \frac{1}{4}\sin\beta$. Additionally, equations (3), (4), (6), (10), (11), and (15) with (28) and (31), gives us $(t_0 + t_1)(u_0 - u_1) = 0$.

In summary, we have the following lemma.

Lemma 2. *Suppose* $m(x, y)$ *is a trigonometric polynomial written in its polyphase form as in equation (1). If* $m(x, y)$ *satisfies properties (i), (ii), (iii), and (iv) with* $M = 1$, *then the coefficients satisfy*

$$a_0 + a_4 + a_8 + a_{12} = \frac{1}{2}(r_0 + r_1), \quad a_2 + a_6 + a_{10} + a_{14} = \frac{1}{2}(r_0 - r_1)$$
$$a_1 + a_5 + a_9 + a_{13} = \frac{1}{2}(s_0 + s_1), \quad a_3 + a_7 + a_{11} + a_{15} = \frac{1}{2}(s_0 - s_1),$$
$$a_0 + a_1 + a_2 + a_3 = \frac{1}{2}(t_0 + t_1), \quad a_8 + a_9 + a_{10} + a_{11} = \frac{1}{2}(t_0 - t_1)$$
$$a_4 + a_5 + a_6 + a_7 = \frac{1}{2}(u_0 + u_1), \quad a_{12} + a_{13} + a_{14} + a_{15} = \frac{1}{2}(u_0 - u_1),$$

and

$$(r_0 + r_1)(s_0 - s_1) = 0$$
$$(t_0 + t_1)(u_0 - u_1) = 0.$$

where r_0, s_0, t_0, u_0 *are given in Lemma 1 and* $r_1 = \frac{1}{4}\cos\alpha$, $s_1 = \frac{1}{4}\sin\alpha$, $=$ $t_1 = \frac{1}{4}\cos\beta$ *and* $u_1 = \frac{1}{4}\sin\beta$.

§4. Nonexistence of higher vanishing moments

Because there are no symmetric compactly supported tensor product wavelets with more than one vanishing moment, it is natural to ask whether there are any nonseparable symmetric solutions with multiple vanishing moments. Although the question is still open, Theorem 3 says their are no symmetric solutions with higher vanishing moments for $m(x, y)$ with $N = 7$. To see this, recall that property (iv) with $M = 2$ implies $\frac{\partial}{\partial x} m(-1, y) = 0$, that is,

$$\begin{bmatrix} 0 \\ 1 \\ -2 \\ 3 \\ -4 \\ 5 \\ -6 \\ 7 \end{bmatrix}^T \begin{bmatrix} a_0 & b_0 & a_1 & b_1 & a_2 & b_2 & a_3 & b_3 \\ b_{15} & a_{15} & b_{14} & a_{14} & b_{13} & a_{13} & b_{12} & a_{12} \\ a_4 & b_4 & a_5 & b_5 & a_6 & b_6 & a_7 & b_7 \\ b_{11} & a_{11} & b_{10} & a_{10} & b_9 & a_9 & b_8 & a_8 \\ a_8 & b_8 & a_9 & b_9 & a_{10} & b_{10} & a_{11} & b_{11} \\ b_7 & a_7 & b_6 & a_6 & b_5 & a_5 & b_4 & a_4 \\ a_{12} & b_{12} & a_{13} & b_{13} & a_{14} & b_{14} & a_{15} & b_{15} \\ b_3 & a_3 & b_2 & a_2 & b_1 & a_1 & b_0 & a_0 \end{bmatrix} \begin{bmatrix} 1 \\ y \\ y^2 \\ y^3 \\ y^4 \\ y^5 \\ y^6 \\ y^7 \end{bmatrix} = 0.$$

Thus,

$$7b_3 - 6a_{12} + 5b_7 - 4a_8 + 3b_{11} - 2a_4 + b_{15} = 0 \tag{61}$$
$$7a_3 - 6b_{12} + 5a_7 - 4b_8 + 3a_{11} - 2b_4 + a_{15} = 0 \tag{62}$$
$$7b_2 - 6a_{13} + 5b_6 - 4a_9 + 3b_{10} - 2a_5 + b_{14} = 0 \tag{63}$$
$$7a_2 - 6b_{13} + 5a_6 - 4b_9 + 3a_{10} - 2b_5 + a_{14} = 0 \tag{64}$$
$$7b_1 - 6a_{14} + 5b_5 - 4a_{10} + 3b_9 - 2a_6 + b_{13} = 0 \tag{65}$$
$$7a_1 - 6b_{14} + 5a_5 - 4b_{10} + 3a_9 - 2b_6 + a_{13} = 0 \tag{66}$$
$$7b_0 - 6a_{15} + 5b_4 - 4a_{11} + 3b_8 - 2a_7 + b_{12} = 0 \tag{67}$$
$$7a_0 - 6b_{15} + 5a_4 - 4b_{11} + 3a_8 - 2b_7 + a_{12} = 0. \tag{68}$$

Similarly, $\frac{\partial}{\partial y} m(x, -1)$ implies

$$7b_3 - 6a_3 + 5b_2 - 4a_2 + 3b_1 - 2a_1 + b_0 = 0 \tag{69}$$
$$7a_{12} - 6b_{12} + 5a_{13} - 4b_{13} + 3a_{14} - 2b_{14} + a_{15} = 0 \tag{70}$$
$$7b_7 - 6a_7 + 5b_6 - 4a_6 + 3b_5 - 2a_5 + b_4 = 0 \tag{71}$$
$$7a_8 - 6b_8 + 5a_9 - 4b_9 + 3a_{10} - 2b_{10} + a_{11} = 0 \tag{72}$$
$$7b_{11} - 6a_{11} + 5b_{10} - 4a_{10} + 3b_9 - 2a_9 + b_8 = 0 \tag{73}$$
$$7a_4 - 6b_4 + 5a_5 - 4b_5 + 3a_6 - 2b_6 + a_7 = 0 \tag{74}$$
$$7b_{15} - 6a_{15} + 5b_{14} - 4a_{14} + 3b_{13} - 2a_{13} + b_{12} = 0 \tag{75}$$
$$7a_0 - 6b_0 + 5a_1 - 4b_1 + 3a_2 - 2b_2 + a_3 = 0. \tag{76}$$

Theorem 3. *Suppose $m(x, y)$ is a trigonometric polynomial written in its polyphase form as in equation (1). If $m(x, y)$ satisfies properties (i), (ii), and (iii), then $m(x, y)$ does not satisfy property (iv) for $M \geq 2$.*

Proof: Adding equations (62), (64), (66), and (68) we have

$$\begin{aligned}
&7(a_0 + a_1 + a_2 + a_3) + 5(a_4 + a_5 + a_6 + a_7) \\
&+ 3(a_8 + a_9 + a_{10} + a_{11}) + (a_{12} + a_{13} + a_{14} + a_{15}) \\
&= 6(b_{12} + b_{13} + b_{14} + b_{15}) + 4(b_8 + b_9 + b_{10} + b_{11}) \\
&+ 2(b_4 + b_5 + b_6 + b_7).
\end{aligned} \tag{77}$$

Substituting Lemma 2, we have

$$\begin{aligned}
7(t_0 + t_1) + 3(u_0 + u_1) &= (t_0 - t_1) + 5(u_0 - u_1) \\
6t_0 - 2u_0 + 8t_1 + 8u_1 &= 0 \\
6t_0 - 2(\frac{1}{4} - t_0) + 8t_1 + 8u_1 &= 0 \\
t_0 + t_1 + u_1 &= \frac{1}{16}.
\end{aligned} \tag{78}$$

Because of the relationships $t_0 + u_0 = 1/4$, $t_0 u_0 = 0, = t_1^2 + u_1^2 = 1/16$, and the constraint $(t_0 + t_1)(u_0 - u_1) = 0$, we only have four cases to consider. They are

case 1: $t_0 = \frac{1}{4}, u_0 = 0, t_1 = \frac{1}{4}$, and $= u_1 = 0$
case 2: $t_0 = \frac{1}{4}, u_0 = 0, t_1 = -\frac{1}{4}$, and $= u_1 = 0$
case 3: $t_0 = 0, u_0 = \frac{1}{4}, t_1 = 0$, and $= u_1 = \frac{1}{4}$
case 4: $t_0 = 0, u_0 = \frac{1}{4}, t_1 = 0$, and $= u_1 = -\frac{1}{4}$.

None of these cases satisfy equation (78). Thus, their = are no solutions with two vanishing moments.

A similar problem arises by adding equations (70), (72), (74), and (76) yielding the inconsistent constraint $r_0 + r_1 + s_1 = \frac{1}{16}$. □

Although the bivariate case allows enough freedom to generate a family of symmetric solutions, it does not allow for multiple vanishing moments at least for the support size we have considered. So, compact support, orthogonality, vanishing moments, and symmetry are again at odds in the construction of bivariate wavelets.

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