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Note Title 9/27/2010

10.2 Runge-Kutta Methods

Taylor Series for $f(x,y)$

Recall

$$f(x+h) = f(x) + h\frac{\partial}{\partial x}f(x) + \frac{h^2}{2!}\frac{\partial^2}{\partial x^2}f(x) + \frac{h^3}{3!}\frac{\partial^3}{\partial x^3}f(x) + \dots$$

$$= \sum_{i=0}^{\infty} \frac{1}{i!}\left(h\frac{\partial}{\partial x}\right)^i f(x)$$

The Taylor series in two variables is

$$f(x+h,\, y+k) = \sum_{i=0}^{\infty} \frac{1}{i!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{i} f(x,y)$$

Note that

$$\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)$$

$$h^2\frac{\partial^2}{\partial x^2} + hk\frac{\partial^2}{\partial x\partial y}$$

$$kh\frac{\partial^2}{\partial y\partial x} + k^2\frac{\partial^2}{\partial y^2}$$

$$\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{0} f(x,y) = f$$

$$\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{1} f(x,y) = h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}$$

$$\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{2} f(x,y) = h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x\partial y} + k^2\frac{\partial^2 f}{\partial y^2}$$

$$\vdots$$

etc.

where f and all partial derivatives are evaluated at (x,y).

<u>Notation</u>

$$f_x = \frac{\partial f}{\partial x} \qquad f_t = \frac{\partial f}{\partial t} \qquad f_{xx} = \frac{\partial^2 f}{\partial x^2} \qquad f_{xt} = \frac{\partial^2 f}{\partial t \partial x}$$

Our functions are such that $f_{xt} = f_{tx}$.

Hence

$$\begin{aligned} f(x+h,\, y+k) = {} & f + (h f_x + k f_y) \\ & + \frac{1}{2}\left(h^2 f_{xx} + 2hk\, f_{xy} + k^2 f_{yy}\right) \\ & + \frac{1}{3!}\left(h^3 f_{xxx} + 3h^2 k\, f_{xxy} + 3hk^2 f_{xyy} + k^3 f_{yyy}\right) \end{aligned}$$

$(a+b)^1$ $\quad + \dots$ $\qquad 1 \quad 1$ $\qquad \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 f$

$(a+b)^2$ $\qquad 1 \quad 2 \quad 1$

$(a+b)^3$ $\qquad 1 \quad 3 \quad 3 \quad 1$

Runge-Kutta method of order 2

We want to solve the IVP

$$\begin{cases} x' = f(t,x) \\ x(a) = b \end{cases}$$

We seek a formula of the form

$$x(t+h) = x(t) + w_1 k_1 + w_2 k_2 \qquad (*)$$

where $k_1 = hf(t,x)$ and $k_2 = hf(t+\alpha h, x+\beta k_1)$

The goal is to determine constants w_1, w_2, α and β so that equation (*) is as accurate as possible.

We would like to reproduce as many terms as possible in the Taylor series

$$x(t+h) = x(t) + h x'(t) + \frac{h^2}{2!} x''(t) + \frac{h^3}{3!} x'''(t) + \cdots \quad (**)$$

Now

$$x(t+h) = x(t) + w_1 h f(t,x) + w_2 h f\big(t+\alpha h,\, x+\beta h f(t,x)\big) \quad (***)$$

We can force (**) and (***) to agree up through the term in h by letting $w_2 = 0$ and $w_1 = 1$

But this is just Euler's method and its order is 1.

To get a higher order method, let us seek an agreement up through the h^2 term.

using the two variable Taylor series

$$f(t + \alpha h, x + \beta h f(t,x))$$

$$= f(t,x) + \alpha h f_t + \beta h f f_x + \frac{1}{2}\left(\alpha h \frac{\partial}{\partial t} + \beta h f \frac{\partial}{\partial x}\right)^2 f(\bar{t},\bar{x})$$

Equation (***) becomes

$$x(t+h) = x(t) + w_1 h f(t,x) + w_2 h \Big[f + \alpha h f_t + \beta h f f_x$$
$$+ \underbrace{\tfrac{1}{2}\Big(\alpha h \tfrac{\partial}{\partial t} + \beta h f \tfrac{\partial}{\partial x}\Big)^2 f(t,x)}\Big]$$

$$= x(t) + w_1 h f + w_2 h f + w_2 h^2 \alpha f_t + w_2 h^2 \beta f f_x + O(h^3)$$

$$= x(t) + (w_1 + w_2) h f + w_2 (\alpha f_t + \beta f f_x) h^2 + O(h^3) \qquad (4)$$

Before comparing (4) to (**) let us rewrite (**) using the following facts.

$$x' = f(t, x)$$

$$x'' = \frac{d}{dt} x'(t)$$

$$= \frac{d}{dt}\left(f(t,x)\right)$$

$$= f_t + \frac{\partial f}{\partial x} \cdot \frac{dx}{dt}$$

$$= f_t + f_x x'$$

$$= f_t + f_x f$$

so (**) becomes

$$x(t+h) = x(t) + hf + \frac{1}{2}h^2\left(f_t + f_x f\right) + O(h^3) \quad \cdots (5)$$