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Note Title 10/29/2010

12.2 Orthogonal Systems and Chebyshev Polynomials

Consider a set of $m+1$ data points

x	x_0	x_1	x_2	$\cdots$	x_m
y	y_0	y_1	y_2	$\cdots$	y_m

Suppose that the data conforms to a relationship

$$y = \sum_{j=0}^{n} c_j g_j(x)$$

In which the functions $g_0, g_1, g_2, \ldots, g_n$ (called basis functions) are known and held fixed.

The coefficients $c_0, c_1, \dots, c_n$ are to be determine according to the least-squares principle.

We can think of the least-squares problem as the set of all functions g that can be expressed as a linear combination of $g_0, g_1, \dots, g_n$ is a vector space G.

i.e.

$$G = \left\{ g : \text{there exist } c_0, c_1, \dots, c_n \text{ such that } g(x) = \sum_{j=0}^{n} c_j \, g_j(x) \right\}$$

The function being sought in the least squares problem is thus an element of the vector space G.

① Question: What basis for G should be chosen for numerical work?

Recall that in the least-squares method, we define an expression

$$\varphi(c_0, c_1, \ldots, c_n) = \sum_{k=0}^{m} \left[\sum_{j=0}^{n} c_j g_j(x_k) - y_k \right]^2$$

and select coefficients to make it as small as possible.

Necessary conditions for minimum

$$\frac{\partial \varphi}{\partial c_i} = 0 \qquad (0 \le i \le n)$$

i.e.

$$\frac{\partial \varphi}{\partial c_i} = \sum_{k=0}^{m} 2\left[\sum_{j=0}^{n} c_j g_j(x_k) - y_k\right] g_i(x_k) = 0 \qquad (0 \le i \le n)$$

Hence rearrange to get

$$\sum_{k=0}^{m}\left[\sum_{j=0}^{n} c_j g_j(x_k)\right] g_i(x_k) = \sum_{k=0}^{m} y_k\, g_i(x_k)$$

i.e.

$$\sum_{j=0}^{n}\left[\sum_{k=0}^{m} g_j(x_k)\, g_i(x_k)\right] c_j = \sum_{k=0}^{m} y_k\, g_i(x_k).$$

which are the normal equations.

The problem reduces to solving the system of normal equations, which depend on the basis $\{g_0, g_1, \dots, g_n\}$.

Required good (nice) properties:

- want equations to be easily solved
- or be capable of being accurately solved.

Suppose that the basis $\{g_0, g_1, \dots, g_n\}$ is orthonormal. the

$$\sum_{k=0}^{m} g_i(x_k)\, g_j(x_k) = \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

The system of normal equations reduces to

$$c_j = \sum_{k=0}^{m} y_k \, g_j(x_k) \qquad (0 \le j \le n)$$

which is an explicit formula for the coefficients c_j.

<u>Example</u>

Consider the space G of all polynomials of degree $\le n$.
A basis of G would be

$$g_0(x) = 1 \quad g_1(x) = x \quad g_2(x) = x^2 \quad \cdots \quad g_n(x) = x^n$$

A typical element of the space G would be of the form

$$g(x) = \sum_{j=0}^{n} c_j\, g_j(x)$$

$$= \sum_{j=0}^{n} c_j\, x^j$$

$$= c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$$

Even though this basis seems <u>natural</u>, it is more often than not a poor choice for numerical work!

(look at graphs of basis functions – very similar).