Note Title 10/25/2010

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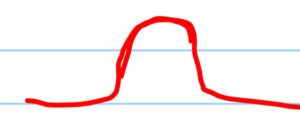
l_2- approximation : we minimize

$$\varphi(a,b) = \sum_{k=0}^{m} (a x_k + b - y_k)^2$$



If the error follow a normal probability distribution, then the minimization of φ produces a best estimate of a and b.

using Calculus, at the minimum we have

$$\frac{\partial \varphi}{\partial a} = 0 \quad \text{and} \quad \frac{\partial \varphi}{\partial b} = 0$$

that is

$$\sum_{k=0}^{m} 2(ax_k + b - y_k)x_k = 0$$

$$\sum_{k=0}^{m} 2(ax_k + b - y_k)1 = 0$$

which are called the <u>normal</u> <u>equations</u>

Rewrite as

$$\begin{cases} \left(\sum_{k=0}^{m} x_k^2\right) a + \left(\sum_{k=0}^{m} x_k\right) b = \sum_{k=0}^{m} y_k x_k \\ \left(\sum_{k=0}^{m} x_k\right) a + \underbrace{\left(\sum_{k=0}^{m} 1\right)}_{(m+1)} b = \sum_{k=0}^{m} y_k \end{cases}$$

If we let $p = \sum_{k=0}^{m} x^2$, $q = \sum_{k=0}^{m} y_k$

$$r = \sum_{k=0}^{m} x_k y_k, \quad s = \sum_{k=0}^{m} x_k^2$$

then the normal equations can be written as

$$\begin{cases} sa + pb = r \\ pa + (m+1)b = q \end{cases}$$

$$\begin{bmatrix} s & p \\ p & (m+1) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} r \\ q \end{bmatrix}$$

which is a 2×2 linear system.

(Easy to solve!)

Example

Find the linear least-squares solution for the following table of values

x	4	7	11	13	17
y	2	0	2	6	7

Plot the original data points and the line.

$$p = \sum_{k=0}^{4} x_k = 4 + 7 + 11 + 13 + 17 = 52$$

$$q = \sum_{k=0}^{4} y_k = 2 + 0 + 2 + 6 + 7 = 17$$

x	y	xy	x^2
4	2	8	16
7	0	0	49
11	2	22	121
13	6	78	169
17	7	119	289
52	17	227	644

$p = \sum_{k=0}^{4} x_k = 52$ $\quad q = \sum_{k=0}^{4} y_k = 17$ $\quad r = \sum_{k=0}^{4} x_k y_k = 227$

$s = \sum_{k=0}^{4} x_k^2 = 644$

Hence

$$\begin{bmatrix} 644 & 52 \\ 52 & 5 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 227 \\ 17 \end{bmatrix}$$

Metho 1: Crammer's Rule

$$a = \frac{\det\begin{bmatrix} 227 & 52 \\ 17 & 5 \end{bmatrix}}{\det\begin{bmatrix} 644 & 52 \\ 52 & 5 \end{bmatrix}} = \frac{(227)(5) - (52)(17)}{(644)(5) - (52)^2} = \frac{251}{516} \approx \boxed{0.486}$$

$$b = \frac{\det\begin{bmatrix} 644 & 227 \\ 52 & 17 \end{bmatrix}}{\det\begin{bmatrix} 644 & 52 \\ 52 & 5 \end{bmatrix}} = \frac{-856}{516} \approx \boxed{-1.659}$$

Best fit $\boxed{y = 0.486x - 1.659}$

Example

What straight line best fits the following data

x	1	2	3	4
y	0	1	1	2

in the least-squares sense?

x	y	xy	x^2
1	0	0	1
2	1	2	4
3	1	3	9
4	2	8	16
10	4	13	30
$p=$	q	$r=$	$s=$

$$\begin{cases} sa + pb = r \\ pa + (m+1)b = q \end{cases}$$

$$\begin{bmatrix} 30 & 10 \\ 10 & 4 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 13 \\ 4 \end{bmatrix}$$

$$a = \frac{\det\begin{bmatrix} 13 & 10 \\ 4 & 4 \end{bmatrix}}{\det\begin{bmatrix} 30 & 10 \\ 10 & 4 \end{bmatrix}} = \frac{(52-40)}{120-100} = \frac{12}{20} = \frac{3}{5}$$

$$b = \frac{\det\begin{bmatrix} 30 & 13 \\ 10 & 4 \end{bmatrix}}{\det\begin{bmatrix} 30 & 10 \\ 10 & 4 \end{bmatrix}} = \frac{120-130}{120-100} = \frac{-10}{20} = -\frac{1}{2}$$

Least squares line :

$$\boxed{y = \frac{3}{5}x - \frac{1}{2}}$$

Another way of looking at the problem.

We want to determine the equation of a line of the form

$$y = ax + b$$

that fits the data best in the least squares sense.

Suppose we have four data points (x_i, y_i) $i = 1, 2, 3, 4$

we get four equations

$$y_1 = ax_1 + b$$
$$y_2 = ax_2 + b$$
$$y_3 = ax_3 + b$$
$$y_4 = ax_4 + b$$

Rewrite as

$$\underbrace{\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ x_3 & 1 \\ x_4 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{\bar{x}} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}}_{\vec{b}}$$

In general we want to solve a linear system

$$A\vec{x} = \vec{b}$$

where A is an $m \times n$ matrix and $m > n$

The solution coincides with the solution of the normal equations

$$A^T A \vec{x} = A^T b.$$

This corresponds to minimizing

$$\| A\vec{x} - \vec{b} \|_2^2$$