# 14 Boundary value problems for ODE:s.

Note Title 11/5/2010

A two-point boundary value problem on the interval $[a,b]$ is one of the form

$$\begin{cases} x''(t) = f(t, x(t), x'(t)) \\ x(a) = \alpha \qquad x(b) = \beta \end{cases} \qquad (1)$$

In this case, the solution function is prescribed at the endpoints of the interval of interest. The two conditions are known as <u>boundary values</u>.

## Examples

$$\begin{cases} x'' = -x \\ x(0) = 2, \quad x(\pi) = 3 \end{cases}$$

$$\begin{cases} x'' = e^{-2t} - 4x - 4x' \\ x(0) = 1 \qquad x(2) = 0 \end{cases}$$

## 14.1 Shooting method

Recall: that in the case of an initial value problem (IVP) for one to specify a particular solution the values of $x$ and $x'$ would be given at some initial point.

In view of this, one way of solving the BVP (1) is to guess $x'(a)$ and then compute the solution of the resulting IVP up to $t = b$. The hope is that the computed solution agrees with the given boundary value, that is $x(b) = \beta$

Computed solution.

given boundary value

If it does not, we can go back and change the guess $x'(a)$.

<u>Algorithm (Shooting Method)</u>

We compute a function $\varphi(z)$ as follows.

Solve the IVP

$$\begin{cases} x' = f(t, x, x') \\ x(a) = \alpha \quad x'(a) = z \end{cases}$$

In the interval $[a,b]$.

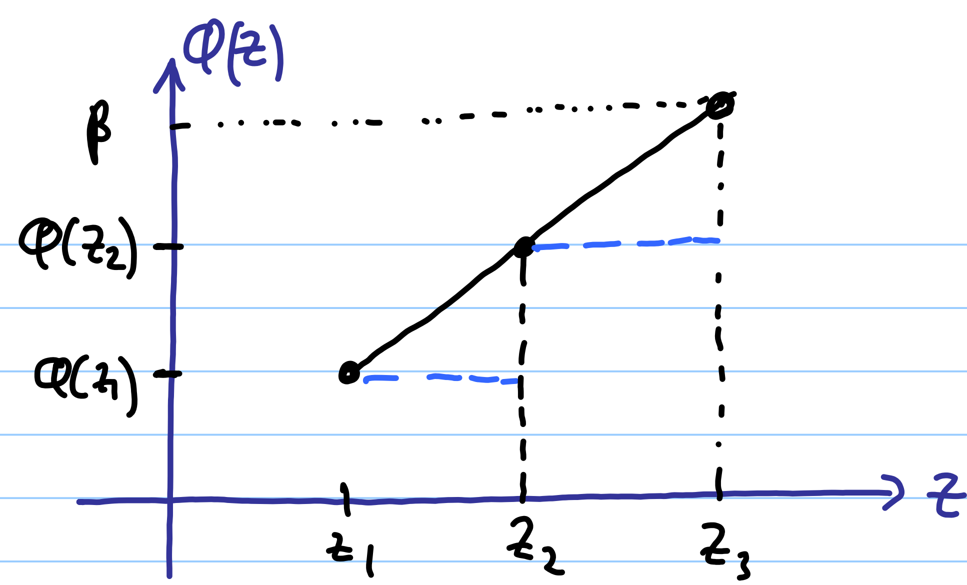
Let $\quad \phi(z) = x(b)$

Objective: adjust $z$ until a value is found for which

$$\phi(z) = \beta.$$

Suppose $z_1$ and $z_2$ are two guesses for the initial condition. So we have $\phi(z_1)$ and $\phi(z_2)$ as the two values of the function at $t=b$.

We derive an estimating formula for the next value $z_3$. Assume that $\phi$ is a linear function.




Using similar triangles

$$\frac{z_3 - z_2}{\beta - \varphi(z_2)} = \frac{z_2 - z_1}{\varphi(z_2) - \varphi(z_1)}$$

Solve for $z_3$

$$z_3 = z_2 + \left[\beta - \varphi(z_2)\right]\left[\frac{z_2 - z_1}{\varphi(z_2) - \varphi(z_1)}\right]$$

Repeat the process to generate the sequence

$$(2) \quad z_{n+1} = z_n + \left[\beta - \varphi(z_n)\right]\left[\frac{z_n - z_{n-1}}{\varphi(z_n) - \varphi(z_{n-1})}\right] \quad n \geq 2$$

Hence to solve the two-point boundary value problem

$$x'' = f(t, x, x') \qquad (3)$$

$$x(a) = \alpha \qquad x(b) = \beta$$

- Solve the IVP

$$\begin{cases} x'' = f(t, x, x') \\ x(a) = \alpha \quad x'(a) = z \end{cases} \qquad (4)$$

from $t=a$ to $t=b$.

denote the solution at $b$ by $\varphi(z)$

- Do this twice for values of $z$ say $z_1$ and $z_2$ and compute $\varphi(z_1)$ and $\varphi(z_2)$

- Calculate a new value of $z$, called $z_3$ using the recursive formula (2).

- Compute $\varphi(z_3)$ by again solving IVP

- Obtain $z_4$ from $z_2$ and $z_3$ in the same way
- Repeat the process

- Stop when $\varphi(z_{n+1}) - \beta$ is small enough!

Example

What is the function $\varphi$ for the two point BVP?

$$\begin{cases} x'' = x \\ x(0) = 1 \quad x(1) = 7 \end{cases}$$

$$x'' = x$$

auxiliary equation : $m^2 = 1$

$m = \pm 1$

$$\therefore x(t) = c_1 e^{m_1 t} + c_2 e^{m_2 t}$$

$$= c_1 e^{t} + c_2 e^{-t}$$ general solution

Given initial condition $x(0) = 1$ $x'(0) = 2$

i.e. IVP $\begin{cases} x'' = x \\ x(0) = 1 \quad x'(0) = 2 \end{cases}$

$$x(0) = c_1 e^{0} + c_2 e^{-0} = 1$$

$$\boxed{c_1 + c_2 = 1}$$

$$x'(t) = c_1 e^{t} - c_2 e^{-t}$$

$$x'(0) = c_1 e^{0} - c_2 e^{-0} = 2$$

$$\boxed{c_1 - c_2 = 2}$$

Solve $\begin{cases} c_1 + c_2 = 1 \\ c_1 - c_2 = 2 \end{cases} \Rightarrow 2c_1 = 1 + 2$

$$c_1 = \frac{1+2}{2}$$

$$c_2 = 1 - c_1$$

$$= 1 - \frac{1+2}{2} = \frac{1-2}{2}$$

$$x(t) = c_1 e^{t} + c_2 e^{-t}$$

$$= \left(\frac{1+z}{2}\right) e^{t} + \left(\frac{1-z}{2}\right) e^{-t}$$

$$\therefore \varphi(z) = x(1)$$

$$= \frac{1+z}{2} e^{1} + \left(\frac{1-z}{2}\right) e^{-1}$$