Note Title 11/3/2010

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Given a table of values (x_k, y_k) $0 \le k \le m$, to obtain a polynomial of degree $\le (n+1)$ that best fits the data, we follow the algorithm (usually $m > n$) [we are using Chebyshev polynomials]

1. Find the smallest interval $[a,b]$ that contains all of the x_k. Thus let $a = \min\{x_k\}$ & $b = \max\{x_k\}$.

2. Make a transformation to the interval $[-1,1]$

by defining

$$z_k = \frac{2x_k - a - b}{b-a} \qquad (0 \le k \le m)$$

3. Decide on the value of n to be used. (Note that 8 or 10 is considered a large value for n).

4. Using Chebyshev polynomials as basis $\left(g(x) = \sum_{j=0}^{n} c_j T_n(x) \right)$ generate the $(n+1) \times (n+1)$ normal equations

$$\sum_{j=0}^{n} \left[\sum_{k=0}^{m} T_i(z_k) T_j(z_k) \right] c_j = \sum_{k=0}^{m} y_k T_i(z_k)$$

$$1 \le i \le n$$

Here let $t_{ij} = T_j(z_k)$ $\quad 0 \le k \le m \quad 0 \le j \le n$

Compute the matrix $T = (t_{jk})$ by the recursive definition

$$T_j(x) = 2x\,T_{j-1}(x) - T_{j-2}(x) \qquad j \geq 2$$

The normal equations have coefficient matrix $A = (a_{ij})_{0:n \times 0:n}$ and right hand side $(b_i)_{0:n}$ given by

$$a_{ij} = \sum_{k=0}^{m} T_i(z_k)\,T_j(z_k) = \sum_{k=0}^{m} t_{ik}\,t_{jk} \qquad 0 \le i,j \le n$$

$$b_i = \sum_{k=0}^{m} y_k\,T_i(z_k) = \sum_{k=0}^{m} y_k\,t_{ik} \qquad 0 \le i \le n$$

5. Use an equation-solving routine to solve the normal equations for the coefficients $c_0, c_1, \ldots, c_n$ in the function

$$g(x) = \sum_{j=0}^{n} c_j T_j(x)$$

6. The polynomial that is being sought is

$$g\left(\frac{2x - a - b}{b - a}\right)$$

<u>Example</u>

x	-1	2	3
y	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{5}{12}$

Find $g(x) = \sum_{j=0}^{n} c_j T_j(x)$, that fits in the least squares sense.

$[a,b] = [-1, 3]$

<u>Step 2:</u> $z_k = \frac{2x_k - a - b}{b-a}$ $\quad 0 \le k \le m$

$$z_0 = \frac{2(x_0) - a - b}{b-a} = \frac{2(-1) - (-1) - 3}{3-(-1)} = -1 \checkmark$$

$$z_1 = \frac{2x_1 - a - b}{b-a} = \frac{2(2) - (-1) - 3}{3-(-1)} = \frac{2}{4} = \frac{1}{2}$$

$$z_2 = \underline{1}$$

Step 3 Take $n = 2 \Rightarrow g(x) = \sum_{j=0}^{n} c_j T_j(x) = \sum_{j=0}^{2} c_j T_j(x)$

Step 4:

$T_0(x) = 1, \quad T_1(x) = x \quad T_2(x) = 2x^2 - 1$

$$a_{ij} = \sum_{k=0}^{m} T_i(z_k) T_j(z_k) \qquad 0 \le i, j \le n.$$

$$a_{11} = \sum_{k=0}^{2} T_i(z_k) T_j(z_k)$$

$$= T_1(-1) T_1(-1) + T_1(\tfrac{1}{2}) T_1(\tfrac{1}{2}) + T_1(1) T_1(1)$$

$$a_{1,0} = \sum_{k=0}^{2} T_1(z_k) T_0(z_k)$$

$$= T_1(-1) T_0(-1) + T_1(\tfrac{1}{2}) T_0(\tfrac{1}{2}) + T_1(1) T_0(1)$$

Applications of Least-Squares

One of the most important applications is the smoothing of data. Smoothing is the fitting of a "smooth" curve to a set of "noisy" values. The values contain experimental errors.

- If one knows the type of function to which the data should conform, then the least-squares procedure can be used to compute any unknown parameters. in the function

- If one simply wishes to smooth the data by fitting them with any convenient function, then polynomials of increasing degree can be used until are reasonable balance between good fit and smoothness is obtained