Boundary Value Problems

Note Title 11/12/2010

Recall: BVP
$$\begin{cases} x'' = f(t, x, x') \\ x(a) = \alpha \qquad x(b) = \beta \end{cases}$$

Discretize
$$\begin{cases} x_0 = \alpha \\ \frac{1}{h^2}\left(x_{i-1} - 2x_i + x_{i+1}\right) = f\left(t_i, x_i, \frac{1}{2h}(x_{i+1} - x_{i-1})\right) \quad 1 \le i \le n-1 \\ x_n = \beta \end{cases}$$

When the RHS is linear, i.e.

$$f(t, x, x') = u(t) + v(t)x + w(t)x'$$

we get

$$\begin{cases} x_0 = \alpha \\ \frac{1}{h^2}\left(x_{i-1} - 2x_i + x_{i+1}\right) = u(t_i) + v(t_i)x_i + w(t_i)\frac{1}{2h}\left(x_{i+1} - x_{i-1}\right) \quad 1 \le i \le n-1 \\ x_n = \beta \end{cases}$$

Rewrite as (on multiplying both sides by h^2)

$$x_{i-1} - 2x_i + x_{i+1} = h^2 u_i + h^2 v_i x_i + w_i \frac{h}{2}\left(x_{i+1} - x_{i-1}\right)$$

$$\Rightarrow \left(1 + \frac{h}{2} w_i\right) x_{i-1} - \left(2 + h^2 v_i\right) x_i + \left(1 - \frac{h}{2} w_i\right) x_{i+1} = h^2 u_i$$

$$\Rightarrow -\left(1+\frac{h}{2}w_i\right)x_{i-1} + \left(2+h^2v_i\right)x_i - \left(1-\frac{h}{2}w_i\right)x_{i+1} = -h^2u_i$$

Let $\begin{cases} a_i = -\left(1+\frac{h}{2}w_i\right) \\ d_i = 2+h^2v_i \\ c_i = -\left(1-\frac{h}{2}w_i\right) \\ b_i = -h^2u_i \end{cases}$ $\quad 0 \le i \le n$

then we get

$$a_i x_{i-1} + d_i x_i + c_i x_{i+1} = b_i$$

Note that when

$i=1$: $\quad a_1x_0 + d_1x_1 + c_1x_2 = b_1$

$\Rightarrow d_1 x_1 + c_1 x_2 = b_1 - a_1 \alpha$ because $x_0 = \alpha$

$i=2: \quad a_2 x_1 + d_2 x_2 + c_2 x_3 = b_2$

$i=3: \quad a_3 x_2 + d_3 x_3 + c_3 x_4 = b_3$

$\vdots$

$i=n-2 \quad a_{n-2} x_{n-3} + d_{n-2} x_{n-2} + c_{n-2} x_{n-1} = b_{n-2}$

$i=n-1 \quad a_{n-1} x_{n-2} + d_{n-1} x_{n-1} + c_{n-1} x_n = b_{n-1}$

$\Rightarrow a_{n-1} x_{n-2} + d_{n-1} x_{n-1} = b_{n-1} - c_{n-1} \beta$ because $x_n = \beta$

Hence we have the tridiagonal system

$$\begin{bmatrix} d_1 & c_1 & & & & \\ a_2 & d_2 & c_2 & & & \\ & a_3 & d_3 & c_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & a_{n-2} & d_{n-2} & c_{n-2} \\ & & & & a_{n-1} & d_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-2} \\ x_{n-1} \end{bmatrix} = \begin{bmatrix} b_1 - a_1\alpha \\ b_2 \\ b_3 \\ \vdots \\ b_{n-2} \\ b_{n-1} - c_{n-1}\beta \end{bmatrix}$$

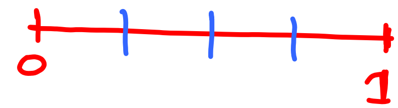
Use routines for solving tridiagonal systems.

<u>Example</u>

Consider the two point BVP

$$\begin{cases} x'' = -x \\ x(0) = 0 \qquad x(1) = 1 \end{cases}$$

Using $h = \frac{1}{4}$ we have that $\quad n = 4$



$t_0 = 0 \quad t_1 = \frac{1}{4} \quad t_2 = \frac{1}{2} \quad t_3 = \frac{3}{4} \quad t_4 = 1$

Recall

$$-\left(1 + \frac{h}{2} w_i\right) x_{i-1} + \left(2 + h^2 v_i\right) x_i - \left(1 - \frac{h}{2} w_i\right) x_{i+1} = -h^2 u_i \qquad 1 \le i \le n-1$$

$f(t, x, x') = -x$

In our case $u(t) = 0, \quad v(t) = -1, \quad w(t) = 0$

So

$$-(1+0) x_{i-1} + \left(2 + \left(\tfrac{1}{4}\right)^2 (-1)\right) x_i - (1 - 0) x_{i+1} = 0 \qquad 1 \le i \le 3$$

i.e.

$$\boxed{-x_{i-1} + \frac{31}{16} x_i - x_{i+1} = 0 \qquad 1 \le i \le 3}$$

So

$$\frac{31}{16}x_1 - x_2 = \underset{\underset{0}{\shortparallel}}{x_0}$$

$$-x_1 + \frac{31}{16}x_2 - x_3 = 0$$

$$-x_2 + \frac{31}{16}x_3 = \underset{\underset{1}{\shortparallel}}{x_4}$$

$$\begin{bmatrix} \frac{31}{16} & -1 & 0 \\ -1 & \frac{31}{16} & -1 \\ 0 & -1 & \frac{31}{16} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Solve the system :

from eq ① $x_1 = \frac{16}{31}x_2$

eqn ② $\quad x_3 = (1 + x_2)\frac{16}{31}$

second eqn $\quad -x_1 + \frac{31}{16}x_2 - x_3 = 0$

$$-\frac{16}{31}x_2 + \frac{31}{16}x_2 - (1+x_2)\frac{16}{31} = 0$$

$$-\frac{32}{31}x_2 + \frac{31}{16}x_2 = \frac{16}{31}$$

$$\boxed{x_2 \approx 0.5701559}$$

$$x_3 = (1 + x_2)\frac{16}{31} \approx 0.810403$$

$$x_1 = \frac{16}{31}x_2 \approx 0.29427$$

$x_0 = 0$, $x_4 = 1$

Errors!! Exact value

$$x'' = -x$$

$m = \alpha + \beta i$

$x(t) = e^{\alpha t}(A\cos\beta t + B\sin\beta t)$

Auxilliary equation: $m^2 = -1$

$$m = \pm i$$

General solution: $x(t) = A\cos t + B\sin t$

$$x(0) = A(1) + B(0) = 0$$

$$A = 0$$

$x(\frac{1}{4}) \approx 0.2940$

$$x(t) = B\sin t$$

$$x(1) = B\sin 1 = 1$$

$$B = \frac{1}{\sin 1}$$

$$\boxed{x(t) = \frac{1}{\sin(1)}\sin(t)}$$