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Note Title 8/23/2010

<u>Example</u>

Consider the Hilbert

$$H_3 = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix}$$

which is ill-conditioned.

Now

$$H_3^{-1} = \begin{bmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{bmatrix}$$

$$\|H_3\|_1 = \|H_3\|_\infty = \frac{11}{6}$$

$$\|H_3^{-1}\|_1 = \|H_3^{-1}\|_\infty = 408$$

Condition number

$$K_1(H_3) = \|H_3\|_1 \cdot \|H_3^{-1}\|_1$$

$$= \frac{11}{6}(408)$$

$$= 748 \quad A \text{ is ill-condition}$$

LARGE

Note

$$K_\infty(H_3) = 748$$

For n as large as 6, the ill-conditioning is extremely bad with

$$\kappa_1(H_6) = \kappa_\infty(H_6) \approx 29 \times 10^6$$

## Basic Iterative Methods

We are still considering the problem of solving $n$ nonsingular equations in $n$ unknowns.

$$Ax = b$$

Suppose that $Q$ and $F$ are $n \times n$ matrices such that

$$A = Q - F$$

Then with the splitting of $A$, the problem may be written as

$$(\Phi - F)x = b$$

i.e.

$$\boxed{\Phi x = Fx + b}$$

Note that $F = \Phi - A$

So

$$\boxed{\Phi x = (\Phi - A)x + b}$$

Algorithm:

- we select a non singular matrix $Q$
- Choose an arbitrary starting vector $x^{(0)}$
- Generate a sequence $x^{(1)}, x^{(2)}, \dots$ recursively

from the equation

$$(*) \qquad Q x^{(k)} = (Q - A) x^{(k-1)} + b \qquad k = 1, 2, 3, \dots$$

In principle, the numerical procedure is designed so that the sequence of vectors converge to the actual solution.

<u>Question</u> How does one choose the non singular matrix $Q$ ?

- System (*) should be easy to solve for $x^{(k)}$ for known right hand side.

Coefficient matrix $Q$ being diagonal, tridiagonal, banded, lower triangular and upper triangular would be nice!

$$2x_1 - x_2 = 3$$
$$x_2 = 5$$

$$\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

- Matrix $Q$ should be chosen to ensure that the sequence $x^{(k)}$ converges irrespective of the initial vector used. (Rapid convergence ideally)

## Jacobi Method

The system $Ax = b$ written is a more detailed form is

$$\begin{array}{l} a_{11}\boxed{x_1} + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}\boxed{x_2} + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}\boxed{x_n} = b_n \end{array}$$

For the ith equation we solve for the ith unknown term. (Assume that all diagonal elements are non zero. Can usually rearrange if not so!)

$$a_{11}x_1 = -(a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n) + b_1$$

$$a_{22}x_2 = -(a_{21}x_1 + a_{23}x_3 + \cdots + a_{2n}x_n) + b_2$$

$$\vdots$$

$$a_{ii}x_i = -(a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{ii-1}x_{i-1} + a_{ii+1}x_{i+1} + \cdots + a_{in}x_n) + b_i$$

$$\vdots$$

$$a_{nn}x_n = -(a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn-1}x_{n-1}) + b_n$$

i.e

$$x_1 = -\frac{1}{a_{11}}\left(a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n\right) + \frac{b_1}{a_{11}}$$

$$x_2 = -\frac{1}{a_{22}}\left(a_{21}x_1 + a_{23}x_3 + \cdots + a_{2n}x_n\right) + \frac{b_2}{a_{22}}$$

$$\vdots$$

$$x_i = -\frac{1}{a_{ii}}\left(a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{ii-1}x_{i-1} + a_{ii-2}x_{i-2} + \cdots + a_{in}x_n\right) + \frac{b_i}{a_{ii}}$$

$$\vdots$$

$$x_n = -\frac{1}{a_{nn}}\left(a_{n1}x_1 + \cdots + a_{nn-1}x_{n-1}\right) + \frac{b_n}{a_{nn}}$$

Compact form:

$$x_1 = -\sum_{j=2}^{n} \frac{a_{1j}}{a_{11}} x_j + \frac{b_1}{a_{11}}$$

$$x_2 = -\sum_{\substack{j=1 \\ j\neq 2}}^{n} \frac{a_{2j}}{a_{22}} x_j + \frac{b_2}{a_{22}}$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

$$x_i = -\sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{a_{ij}}{a_{ii}} x_j + \frac{b_i}{a_{ii}}$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

$$x_n = -\sum_{j=1}^{n-1} \frac{a_{n-1j}}{a_{nn}} x_j + \frac{b_n}{a_{nn}}$$

The Jacobi method is

$$x_1^{(k)} = -\sum_{j=2}^{n} \frac{a_{1j}}{a_{11}} x_j^{(k-1)} + \frac{b_1}{a_{11}}$$

$$x_2^{(k)} = -\sum_{\substack{j=1 \\ j \neq 2}}^{n} \frac{a_{2j}}{a_{22}} x_j^{(k-1)} + \frac{b_2}{a_{22}}$$

$$\vdots$$

$$x_i^{(k)} = -\sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{a_{ij}}{a_{ii}} x_j^{(k-1)} + \frac{b_i}{a_{ii}}$$

$$\vdots$$

$$x_n^{(k)} = -\sum_{j=1}^{n-1} \frac{a_{nj}}{a_{nn}} x_j^{(k-1)} + \frac{b_n}{a_{nn}}$$