MAT 338(01) – Fall 2021 | Murray State University | **Homework 7**
Ordinary Differential Equations | Murray, KY 42071 | **Due October 4, 2021**

Name: ______________________________

Question:	1	2	Total
Points:	32	40	72
Score:			

TO RECEIVE FULL CREDIT YOU MUST SHOW ALL YOUR WORK.

Harvesting a Renewable Resource: Suppose that the population y of a certain species of fish (e.g., tuna or halibut) in a given area of the ocean is described by the logistic equation

$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y.$$

If the population is subjected to harvesting at a rate $H(y,t)$ members per unit time, then the harvested population is modeled by the differential equation

$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y - H(y,t). \tag{1}$$

Although it is desirable to utilize the fish as a food source, it is intuitively clear that if too many fish are caught, then the fish population may be reduced below a useful level and possibly even driven to extinction. The following problems explore some of the questions involved in formulating a rational strategy for managing the fishery.

[32] 1. **Constant Effort Harvesting.** At a given level of effort, it is reasonable to assume that the rate at which fish are caught depends on the population y: the more fish there are, the easier it is to catch them. Thus we assume that the rate at which fish are caught is given by $H(y,t) = Ey$, where E is a positive constant, with units of 1/time, that measures the total effort made to harvest the given species of fish. With this choice for $H(y,t)$, Equation (1) becomes

$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y - Ey. \tag{2}$$

This equation is known as the **Schaefer model** after the biologist M. B. Schaefer, who applied it to fish populations.

(a) Show that if $E < r$, then there are two equilibrium points, $y_1 = 0$ and $y_2 = K\left(1 - \frac{E}{r}\right) > 0$.

(b) Show that $y = y_1$ is unstable and $y = y_2$ is asymptotically stable.

(c) A sustainable yield Y of the fishery is a rate at which fish can be caught indefinitely. It is the product of the effort E and the asymptotically stable population y_2. Find Y as a function of the effort E. The graph of this function is known as the *yield-effort curve.*

(d) Determine E so as to maximize Y and thereby find the **maximum sustainable yield** Y_m .

40 2. **Constant Yield Harvesting.** In this problem, we assume that fish are caught at a constant rate h independent of the size of the fish population, that is, the harvesting rate $H(y,t) = h$. Then y satisfies

$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y - h = f(y). \tag{3}$$

The assumption of a constant catch rate h may be reasonable when y is large but becomes less so when y is small.

(a) If $h < \frac{1}{4}rK$, show that Equation (3) has two equilibrium points y_1 and y_2 with $y_1 < y_2$; determine these points.

(b) Show that $y = y_1$ is unstable and $y = y_2$ is asymptotically stable.

(c) From a plot of $f(y)$ versus y, show that if the initial population $y_0 > y_1$,then $y \to y_2$ as $t \to \infty$, but if $y_0 < y_1$, then y decreases as t increases. Note that $y = 0$ is not an equilibrium point, so if $y_0 < y_1$, then extinction will be reached in a finite time.

(d) If $h > \frac{1}{4}rK$, show that y decreases to zero as t increases regardless of the value of y_0.

(e) If $h = \frac{1}{4}rK$, show that there is a single equilibrium point $y = \frac{1}{2}K$ and that this point is semi-stable. Thus the maximum sustainable yield is $h_m = \frac{1}{4}rK$, corresponding to the equilibrium value $y = \frac{1}{2}K$. Observe that h_m has the same value as Y_m in *Problem 1(d)*. The fishery is considered to be overexploited if y is reduced to a level below $\frac{1}{2}K$.

Earned P: ______