Theorem (3.2.3)

Suppose y_1 and y_2 are solutions to the equation

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

with the initial conditions

$$y(t_0) = y_0, \qquad y'(t_0) = y_0'$$

Then it is always possible to choose constants c_1, c_2 so that

$$y = c_1 y_1(t) + c_2 y_2(t)$$

satisfies the differential equation and initial conditions if and only if the Wronskian

$$W = y_1 y_2' - y_1' y_2 \tag{8}$$

is not zero at the point t_0

Notes

Example

Find the Wronskian for solutions to the differential equation

$$y'' + 5y' + 6y = 0$$

Characteristic equation: $r^2 + 5r + 6 = 0$

$r = -2$ or $r = -3$

Solutions: $y_1 = e^{-2t}$ and $y_2 = e^{-3t}$

$$W(y_1, y_2) = \begin{vmatrix} e^{-2t} & e^{-3t} \\ -2e^{-2t} & -3e^{-3t} \end{vmatrix}$$

$$= -3e^{-5t} + 2e^{-5t} = -e^{-5t}$$

Notes

W is non zero for all values of t
So the functions y_1 and y_2
Can be used to construct solutions of
the differential equation at any
values of t.

Theorem (3.2.4 Fundamental Solutions)

Suppose y_1 and y_2 are solutions to the equation

$$L[y] = y'' + p(t)y' + q(t)y = 0.$$

Then the family of solutions

$$y = c_1y_1 + c_2y_2$$

with arbitrary coefficients c_1, c_2 includes every solution to the differential equation if an only if there is a point t_0 such that

$$W(y_1, y_2)(t_0) \neq 0.$$

The expression $y = c_1y_1 + c_2y_2$ is called the **general solution** of the differential equation above, and in this case y_1 and y_2 are said to form a **fundamental set of solutions** to the differential equation.

Notes

Example

Verify that the functions y_1 and y_2 are solutions of the given differential equation. Do they constitute a fundamental set of solutions?

$$y'' - 2y' + y = 0, \qquad y_1(t) = e^t, \qquad y_2(t) = te^t$$

$y_1 = e^t$:

$y_1'' - 2y_1' + y_1 = e^t - 2e^t + e^t = 0 \Rightarrow y_1$ is a solution

$y_2'' - 2y_2' + y_2 = (2e^t + te^t) - 2(e^t + te^t) + te^t = 0$

$\therefore y_2$ is a solution.

$$W(y_1, y_2)(t) = \begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix} = e^{2t} + te^{2t} - te^{2t} = e^{2t} \neq 0$$ for all values of t

Hence y_1 and y_2 form a fundamental set of solutions.

Notes

Theorem (3.2.5 Existence of Fundamental Set of Solutions)

Consider the differential equation below, whose coefficients p and q are continuous on some open interval I:

$$L[y] = y'' + p(t)y' + q(t)y = 0.$$

Let t_0 *be a point in* I, *and* y_1 *and* y_2 *solutions of the equation with* y_1 *satisfying initial conditions*

$$y_1(t_0) = 1, \qquad y_1'(t_0) = 0,$$

and y_2 *satisfying initial conditions*

$$y_2(t_0) = 0, \qquad y_2'(t_0) = 1.$$

Then y_1, y_2 *form a fundamental set of solutions to the given differential equation.*

In general, a differential equation will have infinitely many different fundamental solution sets. Typically, we pick the one that is most convenient or useful.

Notes

Example

Find the fundamental set specified by Theorem 3.2.5 for the differential equation and initial point

$$y'' + 4y' + 3y = 0, \qquad t_0 = 1$$

Earlier solutions: $y_1 = e^{-t}$ and $y_2 = e^{-3t}$

$$W(y_1,y_2)(t) = \begin{vmatrix} e^{-t} & e^{-3t} \\ -e^{-t} & -3e^{-3t} \end{vmatrix} = -3e^{-4t} + e^{-4t}$$

$$= -2e^{-4t} \neq 0 \text{ for all values of } t$$

So y_1 and y_2 form a fundamental set of solutions.

Notes

But the two solutions do not satisfy the initial condition stated in theorem 3.2.5.

$$y_1(1) = e^{-1} \qquad y_2(1) = e^{-3}$$

$$y_1'(1) = -e^{-1} \qquad y_2'(1) = -3e^{-3}$$

Thus they do not form the fundamental set of solution mentioned in the Theorem.

Let y_3 and y_4 be the fundamental solutions of Theorem 3.2.5

$$y_3(1) = 1, \quad y_3'(1) = 0$$

$$y_4(1) = 0, \quad y_4'(1) = 1$$

Suppose y_1 and y_2 form a fundamental set of solution

$$y_3 = c_1 y_1 + c_2 y_2 \qquad y_3(t_0) = 1, \quad y_3'(t_0) = 0$$

i.e.

$$y_3 = c_1 e^{-t} + c_2 e^{-3t} \qquad y_3(1) = 1, \quad y_3'(1) = 0$$

and

$$y_4 = d_1 e^{-t} + d_2 e^{-3t}, \quad y_4(1) = 0, \quad y_4'(1) = 1$$

Solve:

$$\left.\begin{aligned} y_3 &= c_1 e^{-t} + c_2 e^{-3t} = 1 \\ y_3' &= -c_1 e^{-t} - 3c_2 e^{-3t} = 0 \end{aligned}\right\}. \qquad \left.\begin{aligned} y_4 &= d_1 e^{-t} + d_2 e^{-3t} = 0 \\ y_4' &= -d_1 e^{-t} - 3d_2 e^{-3t} = 1 \end{aligned}\right\}$$

Theorem (3.2.6)

Consider again the equation

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

where p and q are continuous real-valued functions. If $y = u(t) + iv(t)$ is a complex-valued solution of the equation, then its real part u and its imaginary part v are also solutions of this equation.

Notes

Theorem (3.2.7 Abel's Theorem)

Suppose y_1 and y_2 are solutions to the equation

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

where p and q are continuous on some open interval I. Then the $W(y_1, y_2)(t)$ is given by

$$W(y_1, y_2)(t) = ce^{-\int p(t)\,dt}$$

where c is a constant that depends on y_1 and y_2 but not on t. Further, $W(y_1, y_2)(t)$ is either zero for all t in I (if $c = 0$) or else is never zero in I (if $c \neq 0$).

Notes

Example

Use Abel's Theorem to find the Wronskian for solutions to the differential equation

$$y'' + 5y' + 6y = 0$$

From previous work: $y_1 = e^{-2t}$ & $y_2 = e^{-3t}$

Abel's Theorem: $W(y_1, y_2) = ce^{-\int p(t)\,dt} = ce^{-\int 5\,dt} = ce^{-5t}$

From previous work we found that

$W(y_1, y_2) = -e^{-5t}$!

Notes

Notes