Exponential and Logarithmic Functions

5.1 Exponential Functions and Their Graphs

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Definition: Exponential Function

An **exponential function** with **base** b is denoted by

$$f(x) = b^x$$

where b and x are any real numbers such that $b > 0$ and $b \neq 1$.

Examples

Let $f(x) = 3^x$, $g(x) = (\frac{1}{4})^x$ and $h(x) = 10^{x-2}$. Find the following values

(a). $f(2)$ **(b).** $g\left(-\dfrac{3}{2}\right)$ **(c).** $h(2.3)$ **(d).** $f(0)$

Characteristics of Graphs of Exponential Functions

$$f(x) = b^x, \qquad b > 0, \qquad b \neq 1$$

- Domain $(-\infty, \infty)$
- Range $(0, \infty)$
- x-intercepts: none
- y-intercepts: $(0, 1)$
- Horizontal asymptote: x-axis
- The graph passes through $(1, b)$ and $\left(-1, \frac{1}{b}\right)$.
- As x increases, $f(x)$ increases if $b > 1$ and decreases if $0 < b < 1$.
- The function f is one-to-one.

Procedure for Graphing $f(x) = b^x$

- **Step 1:** Label the point $(0, 1)$ corresponding to the y-intercept $f(0)$.
- **Step 2:** Find and label two additional points corresponding to $f(-1)$ and $f(1)$.
- **Step 3:** Connect the three points with a *smooth* curve with the x-axis as the horizontal asymptote.

Example

1. Graph the function $f(x) = 5^x$.
2. Graph the function $f(x) = (\frac{2}{5})^x$.
3. Graph the function $F(x) = 2^{x-1}$. State the domain and range of F.
4. Graph the function $G(x) = 2^x - 1$. State the domain and range of G.

The Natural Base e

A particular irrational number, denoted by the letter e, appears as the base in many applications. It is referred to as the **natural base**.

The exponential function with base e would be written as

$$f(x) = e^x.$$

To nine decimal places we have $e \approx 2.718281828$.

Doubling Time Growth Model

The doubling time growth model is given by

$$P = P_0 2^{t/d}$$

where

- $P =$ Population at time t
- $P_0 =$ Population at time $t = 0$
- $d =$ Doubling time

Example

In 2004, the population of Kazakhstan, a country in Asia, reached 15 million. It is estimated that the population will double in 30 years. If the population continues to grow at the same rate, what will the population be 20 years from now? Round to the nearest million.

The **half-life** of a quantity is the amount of time it takes the quantity to decrease by 50%.

Exponential Decay Model

The exponential decay model with *half life* is given by

$$A = A_0 \left(\frac{1}{2}\right)^{t/h}$$

where

- $A =$ Quantity at time t
- $A_0 =$ Quantity at time $t = 0$
- $h =$ Half-life

Example

The radioactive isotope of potassium ^{42}K which is used in the diagnosis of brain tumors, has a half-life of 12.36 hours. If 500 milligrams of potassium 42 are taken, how many milligrams will remain after 24 hours? Round to the nearest milligram.

Compound Interest

If a **principal** P is invested at an annual **rate** r **compounded** n times a year, then the **amount** A in the account at the end of t years is given by

$$A = P\left(1 + \frac{r}{n}\right)^{nt}.$$

The annual interest rate r is expressed as a decimal.

Example

If \$3,000 is deposited in an account paying 3% compounded quarterly, how much will you have in the account in 7 years?

Continuous Compound Interest

If a **principal** P is invested at an annual **rate** r **compounded continuously**, then the **amount** A in the account at the end of t years is given by

$$A = Pe^{rt}.$$

The annual interest rate r is expressed as a decimal.

Example

If $3,000 is deposited in an account paying 3% compounded continuously, how much will you have in the account in 7 years?